

# Long time existence results for bore-type initial data for BBM-Boussinesq systems

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## Abstract

In this paper we deal with the long time existence for the Cauchy problem associated to BBM-type Boussinesq systems of equations which are asymptotic models for long wave, small amplitude gravity surface water waves. As opposed to previous papers devoted to the long time existence issue, we consider initial data with nontrivial behaviour at infinity which may be used to model bore propagation.

**Keywords** Boussinesq systems, long time existence, bore propagation;

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## 1 Introduction

### 1.1 The *abcd* Boussinesq systems

The following systems of PDEs were introduced in [4] as asymptotic models for studying long wave, small amplitude gravity surface water waves:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + a \varepsilon \operatorname{div} \Delta V + \varepsilon \operatorname{div} (\eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + c \varepsilon \nabla \Delta \eta + \varepsilon V \cdot \nabla V = 0. \end{cases} \quad (1.1)$$

In system (1.1),  $\varepsilon$  is a small parameter while for all  $(t, x) \in [0, \infty) \times \mathbb{R}^n$ :

$$\begin{cases} \eta = \eta(t, x) \in \mathbb{R}, \\ V = V(t, x) \in \mathbb{R}^n. \end{cases}$$

The variable  $\eta$  is an approximation of the deviation of the free surface of the water from the rest state while  $V$  is an approximation of the fluid velocity. The above family of systems is derived from the classical mathematical formulation of the water waves problem. In many applications the water waves problem raises a significant number of difficulties from both a theoretical and a numerical point of view. This is the reason why approximate models have been established, each of them dealing with some particular physical regimes.

The  $abcd$  systems of equations deal with the so called Boussinesq regime which is explained now. Consider a layer of incompressible, irrotational, perfect fluid flowing through a canal with flat bottom represented by the plane:

$$\{(x, y, z) : z = -h\},$$

where  $h > 0$  and assume that the free surface resulting from an initial perturbation of the steady state can be described as being the graph of a function  $\eta$  over the flat bottom. Also, consider the following quantities:  $A = \max_{x,y,t} |\eta|$  the maximum amplitude encountered in the wave motion and  $l$  the smallest wavelength for which the flow has significant energy. Then, the Boussinesq regime is characterized by the following parameters:

$$\alpha = \frac{A}{h}, \quad \beta = \left(\frac{h}{l}\right)^2, \quad S = \frac{\alpha}{\beta}, \quad (1.2)$$

which are supposed to obey the following relations:

$$\alpha \ll 1, \quad \beta \ll 1 \text{ and } S \approx 1.$$

Supposing for simplicity that  $S = 1$  and choosing  $\varepsilon = \alpha = \beta$ , the systems (1.1) are derived back in [4] by a formal series expansion and by neglecting the second and higher order terms. Actually, the zeros on the right hand side of (1.1) can be viewed as the  $O(\varepsilon^2)$  terms neglected in establishing (1.1). The parameters  $a, b, c, d$  are also restricted by the following relation:

$$a + b + c + d = \frac{1}{3}. \quad (1.3)$$

Asymptotic models taking into account general topographies of the bottom were also derived, see [10]. In this situation, one has to furthermore distinguish between two different regimes: small respectively strong topography variations. In [9] time varying bottoms are considered. A systematic study of asymptotic models for the water waves problem along with their rigorous justification can be found for instance in [12]. Let us also point out that the only values of  $n$  for which (1.1) is physically relevant are  $n = 1, 2$ .

The study of the local well-posedness of the  $abcd$  systems is the subject under investigation in several papers beginning with [4] where, besides deriving the family of systems (1.1), it is shown that the linearized equation near the null solution of (1.1) is well posed in two generic cases, namely:

$$a \leq 0, \quad c \leq 0, \quad b \geq 0, \quad d \geq 0 \quad (1.4)$$

$$\text{or } a = c \geq 0 \text{ and } b \geq 0, \quad d \geq 0. \quad (1.5)$$

In [5], the sequel of [4], attention is given to the well-posedness of the nonlinear systems and of some other higher order (formally, more accurate) Boussinesq systems. Other papers that are dealing with the well-posedness of the  $abcd$  systems, for some cases that are not treated in [5], are [2], [6], [11].

The rigorous justification of the fact that systems (1.1) do approximate the water waves problem has been carried on in [7] (see also [12]). In this paper, the authors prove that the error estimate between the solution of (1.1) and the water waves system at time  $t$  is of order  $O(\varepsilon^2 t)$ . It is for this reason that on time scales larger than  $O(\varepsilon^{-2})$  the solutions of (1.1) stop being relevant approximations of the original problem.

The above error estimate result, leads to consider the so-called long time existence<sup>1</sup> problem which we will explain now. First of all, global existence theory of solutions of (1.1) is for the moment not at reach. Indeed, the only global results known are available in dimension 1 for the case:

$$a = b = c = 0, \quad d > 0,$$

which was studied by Amick in [1] and Schonbek in [17] and in the more general case

$$b = d > 0, \quad a \leq 0, \quad c < 0,$$

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<sup>1</sup>abbreviated l.t.e. in the following

which is somehow not satisfactory because one has to assume some smallness condition on the initial data, see [5]. Second of all, as we mentioned above, systems (1.1) are reliable from a practical point of view only on time scales smaller than  $O(\varepsilon^{-2})$  and, as it turns out, on time intervals of order  $O(\varepsilon^{-1})$  the error estimate remains of small order i.e.  $O(\varepsilon)$ . We also mention that on time scales of order  $O(\varepsilon^{-1/2})$ , systems (1.1) behave like the linear wave equation thus, existence results where the solution lives on such time scales are not susceptible of having any predicting features. Also, at time scales of order  $O(\varepsilon^{-1})$  the dispersive and nonlinear effects will have an order one contribution to the wave's evolution.

All the above considerations have led people to consider the l.t.e. problem which consists in constructing solutions of (1.1) for which the maximal time of existence is of formal order  $O(\varepsilon^{-1})$ . Except for the one-dimensional previously mentioned cases, the first long time existence result was obtained in [15] for the case

$$a, c < 0, \quad b, d > 0$$

and for the so called BBM-BBM case corresponding to

$$a = c = 0, \quad b, d > 0. \quad (1.6)$$

The proof adopted in this paper relies on the Nash–Moser theorem and involves a loss of derivatives as well as a relatively high level of regularity. The l.t.e. problem received another satisfactory answer in [16], see also [14], where the case (1.4) was treated and long time existence for the Cauchy problem was systematically proved, provided that the initial data lies in some Sobolev spaces. In [8], we used a different method, from the one applied in [16] and [14], which is based on an energy method applied for spectrally localized equations, in order to obtain l.t.e. results for most of the parameters verifying (1.4). In particular we managed to lower the regularity assumptions needed in order to develop the l.t.e. theory. When considering small variations of the bottom topography the methods presented in [14], [16] and [8] adapt without to much extra effort. Regarding the strongly varying bottoms, l.t.e. results can be found in [13].

The difficulty in obtaining such results comes from the lack of symmetry of (1.1) owing to the  $\varepsilon \eta \operatorname{div} V$  term from the first equation of the  $abcd$ -system. Because of the dispersive operators  $-\varepsilon b \Delta \partial_t + a \operatorname{div} \Delta$ ,  $-\varepsilon d \Delta \partial_t + c \nabla \Delta$ , classical symmetrizing techniques for hyperbolic systems of PDE's are very hard to apply.

The starting point of the present work is the paper [6] where results pertaining to bore propagation are established for the BBM-BBM case, (1.6). The l.t.e. problem has been studied for initial data belonging to Sobolev function spaces  $H^s$  with the index of regularity satisfying (at least):

$$s > \frac{n}{2} + 1.$$

This corresponds to initial perturbation of the rest state that are essentially localized in the space variables. Indeed, when  $s$  is chosen as above,  $H^s$  is embedded in  $\mathcal{C}_0$  the class of continuous bounded functions which vanish at infinity. Because bore-type data manifest non trivial behavior at infinity, one must of course change the functional setting of the initial data. It is exactly this issue that we address in the following: the l.t.e. problem for initial data which can be used to successfully model bore-propagation. As far as we know, these are the first l.t.e. results for data that are outside the Sobolev functional setting. More precisely, the methods that we employ here will allow us to construct solutions of some  $abcd$  systems which live on time intervals of order  $O(\varepsilon^{-1})$  where, for the 1-dimensional case, we might consider general disturbances modeled by continuous functions<sup>2</sup>  $\eta_0^{1D}$  such that:

$$\lim_{x \rightarrow \pm\infty} \eta_0^{1D}(x) = \eta_{\pm}.$$

In the two dimensional setting, the situation in view is that of a 2-dimensional perturbation of the essentially 1-dimensional situation, more precisely we consider:

$$\begin{cases} \eta_0^{2D}(x, y) = \eta_0^{1D}(x) + \phi(x, y), \\ V_0^{2D}(x, y) = (u_0^{1D}(x) + \psi_1(x, y), \psi_2(x, y)) \end{cases}$$

where we ask

$$\lim_{|(x, y)| \rightarrow \infty} |\phi(x, y)| = 0 \text{ and } \lim_{|(x, y)| \rightarrow \infty} |\psi_i(x, y)| = 0 \text{ for } i = 1, 2.$$

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<sup>2</sup>with some extra regularity on its derivative

Let us point out that in some sense, the change of the functional setting in order to study bore propagation corresponds in solving a Neumann problem. Indeed, solving (1.1) with an initial data in  $\mathcal{C}_0$  amounts in solving a Dirichlet-type problem whereas solving it with initial data which is bounded with its gradient belonging to  $\mathcal{C}_0$  amounts to solving a Neumann-type problem.

## 1.2 The main results

In the present paper we will focus our attention on the so called BBM-type Boussinesq systems of equations:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \bar{\eta} + \operatorname{div} \bar{V} + \varepsilon \operatorname{div} (\bar{\eta} \bar{V}) = 0, \\ (I - \varepsilon d \Delta) \partial_t \bar{V} + \nabla \bar{\eta} + \varepsilon \bar{V} \cdot \nabla \bar{V} = 0, \end{cases} \quad (1.7)$$

where the parameters  $b, d$  obey:

$$b, d \geq 0.$$

We address the long time existence problem for the general (1.7) system with initial data that can be used to model bore-type waves. Before stating our results, let us fix the functional framework where we are going to construct our solutions.

We fix two functions  $\chi$  and  $\varphi$  satisfying:

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,$$

(see Proposition 4.1 from the Appendix) and let us denote by  $h$  respectively  $\tilde{h}$  their Fourier inverses. For all  $u \in \mathcal{S}'$ , the nonhomogeneous dyadic blocks are defined as follows:

$$\begin{cases} \Delta_j u = 0 & \text{if } j \leq -2, \\ \Delta_{-1} u = \chi(D) u = \tilde{h} \star u, \\ \Delta_j u = \varphi(2^{-j} D) u = 2^{jd} \int_{\mathbb{R}^n} h(2^j y) u(x - y) dy & \text{if } j \geq 0. \end{cases} \quad (1.8)$$

Let us define now the nonhomogeneous Besov spaces.

**Definition 1.1.** Let  $s \in \mathbb{R}$ ,  $(p, r) \in [1, \infty]$ . The Besov space  $B_{p,r}^s(\mathbb{R}^n)$  is the set of tempered distributions  $u \in \mathcal{S}'$  such that:

$$\|u\|_{B_{p,r}^s} := \left\| \left( 2^{js} \|\Delta_j u\|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

In all that follows, unless otherwise mentioned, the  $j$ -subscript for a tempered distribution is reserved for denoting the frequency localized distribution i.e.:

$$u_j \stackrel{\text{not.}}{=} \Delta_j u. \quad (1.9)$$

**Remark 1.1.** Taking advantage of the Fourier-Plancherel theorem and using (4.9) one sees that the classical Sobolev spaces  $H^s$  coincide with  $B_{2,2}^s$ .

**Remark 1.2.** Let us suppose that  $s \in \mathbb{R}^+ \setminus \mathbb{N}$ . The space  $B_{\infty,\infty}^s$  coincides with the Hölder space  $\mathcal{C}^{[s],s-[s]}$  of bounded functions  $u \in L^\infty$  whose derivatives of order  $|\alpha| \leq [s]$  are bounded and satisfy

$$|\partial^\alpha u(x) - \partial^\alpha u(y)| \leq C |x - y|^{s-[s]} \quad \text{for } |x - y| \leq 1.$$

For all  $s \in \mathbb{R}$ , we define<sup>3</sup>:

$$\begin{cases} s_b = s + \operatorname{sgn}(b), \\ s_d = s + \operatorname{sgn}(d). \end{cases} \quad (1.10)$$

Let us denote by

$$X_{b,d,r}^s(\mathbb{R}^n) = B_{2,r}^{s_b}(\mathbb{R}^n) \times (B_{2,r}^{s_d}(\mathbb{R}^n))^n.$$

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<sup>3</sup>Here, we use the convention  $\operatorname{sgn}(x) = \frac{x}{|x|}$  for  $x \neq 0$  and  $\operatorname{sgn}(0) = 0$ .

For any  $\varepsilon > 0$ , we consider the norm:

$$\left\{ \begin{array}{l} \|(\eta, V)\|_{X_{b,d,r}^{s,\varepsilon}} = \left\| (2^{js} U_j(\eta, V))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} \quad \text{where} \\ U_j^2(\eta, V) = \int_{\mathbb{R}^n} \left( |\eta_j|^2 + \varepsilon b \sum_{k=1,n} |\partial_k \eta_j|^2 + \sum_{k=1,n} |V_j^k|^2 + \varepsilon d \sum_{k,l=1,n} |\partial_l V_j^k|^2 \right). \end{array} \right.$$

The space

$$E_{b,d,r}^s(\mathbb{R}^n) = \left\{ (\eta, V) \in L^\infty(\mathbb{R}^n) \times (L^\infty(\mathbb{R}^n))^n : (\partial_k \eta, \partial_k V) \in X_{b,d,r}^{s-1} \quad \forall k \in \overline{1, n} \right\}$$

endowed with the norm

$$\|(\eta, V)\|_{E_{b,d,r}^{s,\varepsilon}} = \|(\eta, V)\|_{L^\infty} + \left( \sum_{k \in \overline{1, n}} \|(\partial_k \eta, \partial_k V)\|_{X_{b,d,r}^{s-1,\varepsilon}}^2 \right)^{\frac{1}{2}}$$

is a Banach space. An important aspect is that the space  $E_{b,d,r}^s(\mathbb{R})$  admits functions that manifest nontrivial behavior at infinity, see Remark 1.3. Our first result, pertaining to the 1-dimensional case is formulated in the following theorem.

**Theorem 1.1.** *Let us consider  $s \in \mathbb{R}$ ,  $r \in [1, \infty)$  such that  $s > \frac{3}{2}$  or  $s = \frac{3}{2}$  and  $r = 1$ . Let us consider  $(\eta_0, u_0) \in E_{b,d,r}^s(\mathbb{R})$ . Then, there exist two real numbers  $\varepsilon_0$ ,  $C$  both depending on  $s, b, d$ , and on  $\|(\eta_0, u_0)\|_{E_{b,d,r}^{s,1}}$  and a numerical constant  $\tilde{C}$  such that the following holds true. For any  $\varepsilon \leq \varepsilon_0$ , System (1.7) supplemented with the initial data  $(\eta_0, u_0)$ , admits an unique solution  $(\bar{\eta}^\varepsilon, \bar{u}^\varepsilon) \in \mathcal{C}\left([0, \frac{C}{\varepsilon}], E_{b,d,r}^s\right)$ . Moreover, the following estimate holds true:*

$$\sup_{t \in [0, \frac{C}{\varepsilon}]} \|(\bar{\eta}^\varepsilon(t), \bar{u}^\varepsilon(t))\|_{E_{b,d,r}^{s,\varepsilon}} + \sup_{t \in [0, \frac{C}{\varepsilon}]} \|\partial_t \bar{\eta}^\varepsilon(t)\|_{L^\infty} \leq \tilde{C} \|(\eta_0, u_0)\|_{E_{b,d,r}^{s,\varepsilon}}.$$

**Remark 1.3.** Let us give an example of function that fits into our framework but is not covered by previous works dedicated to the long time existence problem. A reasonable initial data for modeling bores would be:

$$\eta_0(x) = \tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

which is smooth and manifests nontrivial behavior at  $\pm\infty$ , i.e.

$$\lim_{x \rightarrow \infty} \eta_0(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \eta_0(x) = -1.$$

One can verify that  $\tanh \in \bigcap_{s \geq 0} E_{b,d,r}^{s,1}$  but does not belong to any Sobolev space  $H^s$ .

Let us consider  $\sigma \geq \frac{1}{2}$ ,  $s \geq 1$  with the convention that whenever there is equality in one of the previous relations then  $r = 1$ . We denote by  $M_{b,d,r}^{\sigma,s}$  the space of  $(\eta, V) \in \mathcal{C}(\mathbb{R}^2) \times (\mathcal{C}(\mathbb{R}^2))^2$  such that there exists  $(\eta^{1D}, V^{1D}) \in E_{b,d,r}^\sigma(\mathbb{R})$  and  $(\eta^{2D}, V^{2D}) \in X_{b,d,r}^s(\mathbb{R}^2)$  for which

$$\left\{ \begin{array}{l} \eta(x, y) = \eta^{1D}(x) + \eta^{2D}(x, y), \\ V(x, y) = (V^{1D}(x) + V_1^{2D}(x, y), V_2^{2D}(x, y)) \end{array} \right. \quad (1.11)$$

for all  $(x, y) \in \mathbb{R}^2$ . Let us introduce  $i : E_{b,d,r}^\sigma(\mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R}^2) \times (\mathcal{C}(\mathbb{R}^2))^2$  such that for all  $(x, y) \in \mathbb{R}^2$  we have:

$$i((\eta, u))(x, y) = (\eta(x), (u(x), 0)).$$

Of course, because of the fact  $i(E_{b,d,r}^\sigma(\mathbb{R})) \cap X_{b,d,r}^s(\mathbb{R}^2) = \{0\}$ , the functions appearing in the decomposition (1.11) are unique. For all  $\varepsilon > 0$ , let us consider on  $M_{b,d,r}^{\sigma,s}$ , the norm

$$\|(\eta, V)\|_{M_{b,d,r}^{\sigma,s,\varepsilon}} = \|(\eta^{1D}, V^{1D})\|_{E_{b,d,r}^{\sigma,\varepsilon}(\mathbb{R})} + \|(\eta^{2D}, V^{2D})\|_{X_{b,d,r}^{s,\varepsilon}(\mathbb{R}^2)}.$$

It is easy to see that  $(M_{b,d,r}^{\sigma,s}, \|\cdot\|_{M_{b,d,r}^{\sigma,s,\varepsilon}})$  is a Banach space. We can now formulate the result pertaining to the 2-dimensional case:

**Theorem 1.2.** *Let us consider  $s, \sigma \in \mathbb{R}$ ,  $r \in [1, \infty)$  such that*

$$s > 2 \text{ or } s = 2 \text{ and } r = 1. \quad (1.12)$$

and

$$\sigma > s + \frac{3}{2} \text{ and } \sigma - \frac{1}{2} \in \mathbb{R} \setminus \mathbb{N}. \quad (1.13)$$

*Let us consider  $(\eta_0, u_0) \in E_{b,d,r}^\sigma(\mathbb{R})$  and  $(\phi, \psi) \in X_{b,d,r}^s(\mathbb{R}^2)$ . Then, there exist two real numbers  $\varepsilon_0$ ,  $C$  both depending on  $s, b, d$ , and on  $\|(\eta_0, u_0)\|_{E_{b,d,r}^{s,1}} + \|(\phi, \psi)\|_{X_{b,d,r}^{s,1}(\mathbb{R}^2)}$  and a numerical constant  $\tilde{C}$  such that the following holds true. For any  $\varepsilon \leq \varepsilon_0$ , system (1.7) supplemented with the initial data*

$$\begin{cases} \bar{\eta}_0(x, y) = \eta_0(x) + \phi(x, y), \\ \bar{V}_0(x, y) = (u_0(x), 0) + \psi(x, y), \end{cases}$$

*admits an unique solution  $(\bar{\eta}^\varepsilon, \bar{V}^\varepsilon) \in \mathcal{C}\left([0, \frac{C}{\varepsilon}], M_{b,d,r}^{\sigma,s}\right)$ . Moreover, the following estimate holds true:*

$$\sup_{t \in [0, \frac{C}{\varepsilon}]} \left( \|(\bar{\eta}^\varepsilon(t), \bar{V}^\varepsilon(t))\|_{M_{b,d,r}^{\sigma,s,\varepsilon}} + \sup_{t \in [0, \frac{C}{\varepsilon}]} \|\partial_t \bar{\eta}^\varepsilon(t)\|_{L^\infty} \right) \leq \tilde{C} \left( \|(\eta_0, u_0)\|_{E_{b,d,r}^{\sigma,\varepsilon}} + \|(\phi, \psi)\|_{X_{b,d,r}^{s,\varepsilon}} \right)$$

The results presented in Theorems 1.1, 1.2 are a by-product of a general result that we obtain later in the paper (see Theorem 2.1). The method of proof consists of conveniently splitting the initial data into two parts and of performing energy estimates on a slightly more general system than (1.7). The estimates are a refined version of those obtained in [8] and they allow us to handle the fact that the bore type functions do not belong to  $L^2$ .

### 1.3 Notations

Let us introduce some notations. For any vector field  $U : \mathbb{R}^n \rightarrow \mathbb{R}^n$  we denote by  $\nabla U : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  and by the  $n \times n$  matrices defined by:

$$(\nabla U)_{ij} = \partial_i U^j,$$

In the same manner we define  $\nabla^2 U : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$  as:

$$(\nabla^2 U)_{ijk} = \partial_{ij}^2 U^k.$$

We will suppose that all vectors appearing are column vectors and thus the (classical) product between a matrix field  $A$  and a vector field  $U$  will be the vector<sup>4</sup>:

$$(AU)^i = A_{ij} U^j.$$

We will often write the contraction operation between  $\nabla^2 U$  and a vector field  $V$  by

$$(\nabla^2 U : V)_{ij} = \partial_{ij}^2 U^k V^k$$

If  $U, V : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are two vector fields and  $A, B : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  two matrix fields we denote:

$$\begin{aligned} UV &= U^i V^i, \quad A : B = A_{ij} B_{ij}, \\ \langle U, V \rangle_{L^2} &= \int U^i V^i, \quad \langle A, B \rangle_{L^2} = \int A_{ij} B_{ij} \\ \|U\|_{L^2}^2 &= \langle U, U \rangle_{L^2}, \quad \|A\|_{L^2}^2 = \langle A, A \rangle_{L^2} \\ \|\nabla^2 U\|_{L^2}^2 &= \int \nabla U : \nabla U = \int (\partial_{ij} U^k)^2 \end{aligned}$$

Also, the tensorial product of two vector fields  $U, V$  is defined as the matrix field  $U \otimes V : \mathbb{R}^n \rightarrow \mathcal{M}_n(\mathbb{R})$  given by:

$$(U \otimes V)_{ij} = U^i V^j.$$

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<sup>4</sup>From now on we will use the Einstein summation convention over repeated indices.

## 2 Some intermediate results

The method of proof of the main results naturally leads us to study the following system:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + \varepsilon \operatorname{div} (\eta W_1 + h V + \beta \eta V) = \varepsilon f, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + \varepsilon (W_2 + \beta V) \cdot \nabla V + \varepsilon V \cdot \nabla W_3 = \varepsilon g \\ \eta|_{t=0} = \eta_0, \quad V|_{t=0} = V_0 \end{cases} \quad (\mathcal{S}_\varepsilon(\mathcal{D}))$$

where  $\varepsilon, \beta \in [0, 1]$  and  $\mathcal{D} = (\eta_0, V_0, f, g, h, W_1, W_2, W_3)$ . The above system captures a very general form of weakly dispersive quasilinear systems. Let us consider  $s \in \mathbb{R}$  such that

$$s > \frac{n}{2} + 1 \text{ or } s = \frac{n}{2} + 1 \text{ and } r = 1. \quad (2.1)$$

Let us fix the following notations:

$$\begin{cases} U_j^2(t) = \|(\eta_j(t), V_j(t))\|_{L^2}^2 + \varepsilon \left\| \left( \sqrt{b} \nabla \eta_j(t), \sqrt{d} \nabla V_j(t) \right) \right\|_{L^2}^2, \\ U_s(t) = \left\| (2^{js} U_j(t))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}, \quad U(t) = \|(\eta, \nabla \eta, V, \nabla V)\|_{L^\infty \cap L^{p_2}}, \quad F_s(t) = \|(f(t), g(t))\|_{B_{2,r}^s}, \\ \mathcal{W}_s(t) = \|h(t)\|_{L^\infty} + \|\nabla h(t)\|_{B_{p_1,r}^s} + \|\partial_t h(t)\|_{L^\infty} + \sum_{i=1}^3 \left( \|(W_i(t))\|_{L^\infty} + \|\nabla W_i(t)\|_{B_{p_1,r}^s} \right). \end{cases} \quad (2.2)$$

In order to ease the reading we will rather skip denoting the time dependency in the computations that follow. We are now in the position of stating the following:

**Theorem 2.1.** *Let us consider  $b, d \geq 0$ , two real numbers with  $b + d > 0$ ,  $(p_1, r) \in [2, \infty] \times [1, \infty)$ ,  $p_2$  such that*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$$

and  $s, s_b, s_d > 0$  as defined in (2.1) and (1.10). Let us also consider  $(f, g) \in \mathcal{C}([0, T], B_{2,r}^s \times (B_{2,r}^s)^n)$ ,  $h \in \mathcal{C}([0, T], L^\infty)$  with  $\nabla h \in \mathcal{C}([0, T], B_{p_1,r}^s)$ ,  $\partial_t h \in L^\infty([0, T], L^\infty)$  and for  $i = \overline{1, 3}$  consider the vector fields  $W_i \in \mathcal{C}([0, T], (L^\infty)^n)$  with  $\nabla W_i \in \mathcal{C}([0, T], (B_{p_1,r}^s)^n)$ . Then, for all  $(\eta_0, V_0) \in B_{2,r}^{s_b} \times (B_{2,r}^{s_d})^n$ , writing  $\mathcal{D} = (\eta_0, V_0, f, g, h, W_1, W_2, W_3)$ , there exists a  $\bar{T} \in (0, T]$  such that  $\mathcal{S}_\varepsilon(\mathcal{D})$  admits a unique solution  $(\eta, V) \in \mathcal{C}([0, \bar{T}], B_{2,r}^{s_b} \times (B_{2,r}^{s_d})^n)$  with  $(\partial_t \eta, \partial_t V) \in \mathcal{C}([0, \bar{T}], B_{2,r}^{s-1+2\operatorname{sgn} b} \times (B_{2,r}^{s-1+2\operatorname{sgn} d})^n)$ . Moreover if  $T^* \in \mathbb{R}^+$  is such that

$$\int_0^{T^*} U(\tau) d\tau < \infty, \quad (2.3)$$

then the solution  $(\eta, V)$  can be continued after  $T^*$ .

Of course, in what the existence and uniqueness of solutions of  $\mathcal{S}_\varepsilon(\mathcal{D})$  is concerned, the results presented in Theorem 2.1 are not optimal. However, the aspect that we wish to emphasize in this paper is the possibility of solving the above system on what we named long time scales and with regards to this matter the extra regularity is necessary in order to develop the forthcoming theory. The next result is the main ingredient in proving the l.t.e. results announced in Theorem 1.1 and Theorem 1.2.

**Theorem 2.2.** *Let us fix  $(\eta_0, V_0) \in X_{b,d,r}^s$ . For  $\varepsilon \in (0, 1]$  we consider  $(f^\varepsilon, g^\varepsilon) \in \mathcal{C}([0, T^\varepsilon], B_{2,r}^s \times (B_{2,r}^s)^n)$ ,  $h^\varepsilon \in \mathcal{C}([0, T^\varepsilon], L^\infty)$  with  $\nabla h^\varepsilon \in \mathcal{C}([0, T^\varepsilon], B_{p_1,r}^s)$ ,  $\partial_t h^\varepsilon \in L^\infty([0, T^\varepsilon], L^\infty)$  and for  $i = \overline{1, 3}$  we consider the vector fields  $W_i^\varepsilon \in \mathcal{C}([0, T^\varepsilon], (L^\infty)^n)$  with  $\nabla W_i^\varepsilon \in \mathcal{C}([0, T^\varepsilon], (B_{p_1,r}^s)^n)$ . Assume that*

$$\sup_{\varepsilon \in [0, 1]} \sup_{\tau \in [0, T^\varepsilon]} \mathcal{W}_s^\varepsilon(\tau) < \infty, \quad \sup_{\varepsilon \in [0, 1]} \sup_{\tau \in [0, T^\varepsilon]} F_s^\varepsilon(\tau) < \infty,$$

where

$$\mathcal{W}_s^\varepsilon(t) = \|h^\varepsilon(t)\|_{L^\infty} + \|\nabla h^\varepsilon(t)\|_{B_{p_1,r}^s} + \|\partial_t h^\varepsilon(t)\|_{L^\infty} + \sum_{i=1}^3 \left( \|(W_i^\varepsilon(t))\|_{L^\infty} + \|\nabla W_i^\varepsilon(t)\|_{B_{p_1,r}^s} \right),$$

and

$$F_s^\varepsilon(t) = \|(f^\varepsilon(t), g^\varepsilon(t))\|_{B_{2,r}^s}.$$

We denote by  $\mathcal{D}^\varepsilon = (\eta_0, V_0, f^\varepsilon, g^\varepsilon, h^\varepsilon, W_1^\varepsilon, W_2^\varepsilon, W_3^\varepsilon)$ . Then, there exist two real numbers  $\varepsilon_0, C$  both depending on  $s, n, b, d, \|(\eta_0, V_0)\|_{X_{b,d,r}^{s,1}}, \sup_{\varepsilon \in [0,1]} \sup_{\tau \in [0,T^\varepsilon]} \mathcal{W}_s^\varepsilon(\tau), \sup_{\varepsilon \in [0,1]} \sup_{\tau \in [0,T^\varepsilon]} F_s^\varepsilon(\tau)$  and a numerical constant  $\tilde{C} = \tilde{C}(n)$  such that the following holds true. For any  $\varepsilon \leq \varepsilon_0$ , the maximal time of existence  $T_{\max}^\varepsilon$  of the unique solution  $(\eta^\varepsilon, V^\varepsilon)$  of system  $\mathcal{S}_\varepsilon(\mathcal{D}^\varepsilon)$  satisfies the following lower bound:

$$T_{\max}^\varepsilon \geq T_\star^\varepsilon \stackrel{\text{def.}}{=} \min \left\{ T^\varepsilon, \frac{C}{\varepsilon} \right\}. \quad (2.4)$$

Moreover, we have that:

$$\sup_{t \in [0, T_\star^\varepsilon]} \|(\eta^\varepsilon(t), V^\varepsilon(t))\|_{X_{b,d,r}^{s,\varepsilon}} + \sup_{t \in [0, T_\star^\varepsilon]} \|\partial_t \eta^\varepsilon(t)\|_{L^\infty} \leq \tilde{C}(n) \|(\eta_0, V_0)\|_{X_{b,d,r}^{s,\varepsilon}}. \quad (2.5)$$

Of course, when the data  $\mathcal{D}^\varepsilon$  does not depend on  $\varepsilon$  we obtain the following:

**Corollary 3.** Let us fix  $(\eta_0, V_0) \in X_{b,d,r}^s$  and  $s$  and  $p_1$  as above. Also, consider  $(f, g) \in \mathcal{C}([0, T], B_{2,r}^s \times (B_{2,r}^s)^n)$ ,  $h \in \mathcal{C}([0, T], L^\infty)$  with  $\nabla h \in \mathcal{C}([0, T], B_{p_1,r}^s)$ ,  $\partial_t h \in L^\infty([0, T], L^\infty)$  and for  $i = \overline{1, 3}$  consider the vector fields  $W_i \in \mathcal{C}([0, T], (L^\infty)^n)$  with  $\nabla W_i \in \mathcal{C}([0, T], (B_{p_1,r}^s)^n)$ . We denote by  $\mathcal{D} = (\eta_0, V_0, f, g, h, W_1, W_2, W_3)$ . Then, there exist two real numbers  $\varepsilon_0, C$  both depending on  $s, n, b, d, \|(\eta_0, V_0)\|_{X_{b,d,r}^{s,1}}, \sup_{\tau \in [0,T]} \mathcal{W}_s(\tau), \sup_{\tau \in [0,T]} F_s(\tau)$  and a numerical constant  $\tilde{C} = \tilde{C}(n)$  such that the following holds true. For any  $\varepsilon \leq \varepsilon_0$ , the maximal time of existence  $T_{\max}^\varepsilon$  of the unique solution  $(\eta^\varepsilon, V^\varepsilon)$  of the system  $\mathcal{S}_\varepsilon(\mathcal{D})$  satisfies the following lower bound:

$$T_{\max}^\varepsilon \geq T_\star^\varepsilon \stackrel{\text{def.}}{=} \min \left\{ T^\varepsilon, \frac{C}{\varepsilon} \right\}. \quad (2.6)$$

Moreover, we have that:

$$\sup_{t \in [0, T_\star^\varepsilon]} \|(\eta^\varepsilon(t), V^\varepsilon(t))\|_{X_{b,d,r}^{s,\varepsilon}} + \sup_{t \in [0, T_\star^\varepsilon]} \|\partial_t \eta^\varepsilon(t)\|_{L^\infty} \leq \tilde{C}(n) \|(\eta_0, V_0)\|_{X_{b,d,r}^{s,\varepsilon}}. \quad (2.7)$$

**Remark 2.1.** The choice of  $s$  according to relation (2.1) ensures that we have the following embedding:  $B_{2,r}^s \hookrightarrow L^{p_2}$  and  $B_{2,r}^s \hookrightarrow L^\infty$ . In particular, we also have

$$U(t) \leq_n U_s(t),$$

a fact that will be systematically used in all that follows.

**Remark 2.2.** The explosion criterion (2.3) implies that the life span of the solution does not depend on its possible extra regularity above the critical level  $B_{2,1}^{\frac{n}{2}+1}$ .

The plan of the proof of Theorem 2.1 is the following. First, we derive a priori estimates using a spectral localization of the system  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . Then, we use the so called Friedrichs method in order to construct a sequence of functions that solves a family of ODE's which approximate system  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . Finally, using a compactness method we show that we can construct a solution of the system  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . Theorem 2.2 is obtained using a bootstrap argument.

## 2.1 Proof of Theorem 2.1

### 2.1.1 A priori estimates

First of all we will derive a priori estimates. Thus, let us consider  $(\eta, V) \in \mathcal{C}([0, \bar{T}], B_{2,r}^{s_b} \times (B_{2,r}^{s_d})^n)$  a solution of  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . As announced, we proceed by localizing the system  $(\mathcal{S}_\varepsilon(\mathcal{D}))$  in the frequency space such that we obtain:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta_j + \operatorname{div} V_j + \varepsilon \nabla \eta_j (W_1 + \beta V) + \varepsilon (h + \beta \eta) \operatorname{div} V_j = \varepsilon f_j + \varepsilon R_{1j}, \\ (I - \varepsilon d \Delta) \partial_t V_j + \nabla \eta_j + \varepsilon (W_2 + \beta V) \cdot \nabla V_j = \varepsilon g_j + \varepsilon R_{2j}, \end{cases} \quad (2.8)$$



where

$$\begin{cases} R_{1j} = \beta [V, \Delta_j] \nabla \eta + [W_1, \Delta_j] \nabla \eta + \beta [\eta, \Delta_j] \operatorname{div} V + [h, \Delta_j] \operatorname{div} V - \Delta_j (\nabla h V) - \Delta_j (\eta \operatorname{div} W_1), \\ R_{2j} = [(W_2 + \beta V) \cdot \nabla, \Delta_j] V - \Delta_j (V \cdot \nabla W_3), \end{cases} \quad (2.9)$$

We multiply the first equation of (2.8) with  $\eta_j$  and the second one with  $1 + \alpha \varepsilon (h + \beta \eta) V_j$ , we add the results and by integrating over  $\mathbb{R}^n$  we obtain that:

$$\frac{1}{2} \frac{d}{dt} \left[ \int \eta_j^2 + \varepsilon b |\nabla \eta_j|^2 + (1 + \alpha \varepsilon (h + \beta \eta)) (V_j^2 + \varepsilon d \nabla V_j : \nabla V_j) \right] = \sum_{i=1}^7 T_i$$

where:

$$\begin{aligned} T_1 &= - \int (1 + \varepsilon (h + \beta \eta)) \eta_j \operatorname{div} V_j - \int (1 + \alpha \varepsilon (h + \beta \eta)) \nabla \eta_j V_j, \\ T_2 &= -\varepsilon \int \nabla \eta_j (W_1 + \beta V) \eta_j, \\ T_3 &= -\varepsilon \langle (W_2 + \beta V) \cdot \nabla V_j, (1 + \alpha \varepsilon (h + \beta \eta)) V_j \rangle_{L^2}, \\ T_4 &= \varepsilon \int f_j \eta_j + \varepsilon \int (1 + \alpha \varepsilon (h + \beta \eta)) g_j V_j, \\ T_5 &= \varepsilon \int R_{1j} \eta_j + \varepsilon \int (1 + \alpha \varepsilon (h + \beta \eta)) R_{2j} V_j, \\ T_6 &= -\frac{1}{2} \alpha \varepsilon \langle V_j, (\partial_t h + \beta \partial_t \eta) V_j \rangle_{L^2} - \frac{1}{2} \alpha d \varepsilon^2 \langle \nabla V_j, (\partial_t h + \beta \partial_t \eta) \nabla V_j \rangle_{L^2}, \\ T_7 &= -\alpha d \varepsilon^2 \langle \partial_t \nabla V_j, \nabla (h + \beta \eta) \otimes V_j \rangle_{L^2}. \end{aligned}$$

Here, we use  $\alpha$  as a parameter in order to obtain the desired estimates from a single "strike". Indeed only the values  $\alpha \in \{0, 1\}$  will be of interest to us and, as we will see,  $\alpha = 0$  will give us the estimate necessary to develop the existence theory while  $\alpha = 1$  will lead to an estimate that is the key point in finding the lower bound on the time of existence.

Let us estimate the  $T_i$ 's. Regarding the first term, we write that:

$$\begin{aligned} T_1 &= - \int (1 + \varepsilon (h + \beta \eta)) \eta_j \operatorname{div} V_j - \int (1 + \alpha \varepsilon (h + \beta \eta)) \nabla \eta_j V_j \\ &= \alpha \varepsilon \int (\nabla h + \beta \nabla \eta) \eta_j V_j - \varepsilon (1 - \alpha) \int (\beta \eta + h) \eta_j \operatorname{div} V_j \\ &\leq \alpha \varepsilon C (\|\nabla h\|_{L^\infty} + \beta \|\nabla \eta\|_{L^\infty}) \|\eta_j\|_{L^2} \|V_j\|_{L^2} \\ &\quad + C \frac{(1 - \alpha) \sqrt{\varepsilon}}{\max(\sqrt{b}, \sqrt{d})} U_j^2 (\mathcal{W}_s + \beta U) \\ &\leq \alpha \varepsilon C U_j^2 (\mathcal{W}_s + \beta U) + C \frac{(1 - \alpha) \sqrt{\varepsilon}}{\max(\sqrt{b}, \sqrt{d})} U_j^2 (\mathcal{W}_s + \beta U). \end{aligned} \quad (2.10)$$

Let us bound the second term:

$$\begin{aligned} T_2 &= -\varepsilon \int (W_1 + \beta V) \nabla \eta_j \eta_j \leq \frac{\varepsilon}{2} \int (\|\operatorname{div} W_1\|_{L^\infty} + \beta \|\operatorname{div} V\|_{L^\infty}) \eta_j^2 \\ &\leq \frac{\varepsilon}{2} U_j^2 (\mathcal{W}_s + \beta U). \end{aligned} \quad (2.11)$$

Using the Einstein summation convention over repeated indices, the term  $T_3$  is treated as follows:

$$\begin{aligned} T_3 &= -\varepsilon \langle (W_2 + \beta V) \cdot \nabla V_j, (1 + \alpha \varepsilon (\beta \eta + h)) V_j \rangle_{L^2} \\ &= -\varepsilon \int (1 + \alpha \varepsilon (\beta \eta + h)) (\beta V^m + W_2^m) \partial_m V_j^k V_j^k \end{aligned}$$

$$\begin{aligned}
 &= \frac{\varepsilon}{2} \int \partial_m ((1 + \alpha\varepsilon(\beta\eta + h)) (\beta V^m + W_2^m)) V_j^k V_j^k \\
 &= \frac{\varepsilon}{2} \int \operatorname{div} ((1 + \alpha\varepsilon(\beta\eta + h)) (\beta V + W_2)) |V_j|^2 \\
 &\leq \frac{\varepsilon}{2} U_j^2 (\|W_2\|_{L^\infty} + \beta \|V\|_{L^\infty} + \alpha\varepsilon \|(\beta\eta, \beta\nabla\eta, h, \nabla h)\|_{L^\infty} \|(\beta V, \beta\nabla V, W_2, \nabla W_2)\|_{L^\infty}) \\
 &\leq \frac{\varepsilon}{2} U_j^2 (\mathcal{W}_s + \beta U + \alpha\varepsilon (\mathcal{W}_s + \beta U)^2). \tag{2.12}
 \end{aligned}$$

Next, we bound the term  $T_4$ :

$$\begin{aligned}
 T_4 &= \varepsilon \int f_j \eta_j + \varepsilon \int (1 + \alpha\varepsilon(h + \beta\eta)) g_j V_j \\
 &\leq \varepsilon \|f_j\|_{L^2} \|\eta_j\|_{L^2} + \varepsilon (1 + \alpha\varepsilon \|h\|_{L^\infty} + \alpha\beta\varepsilon \|\eta\|_{L^\infty}) \|g_j\|_{L^2} \|V_j\|_{L^2} \\
 &\leq \varepsilon C 2^{-js} c_j^1(t) U_j F_s (1 + \alpha\varepsilon (\mathcal{W}_s + \beta U)). \tag{2.13}
 \end{aligned}$$

Treating the fifth term is done in the following lines. First we write that:

$$\begin{aligned}
 T_5 &= \varepsilon \int R_{1j} \eta_j + \varepsilon \int (1 + \alpha\varepsilon(h + \beta\eta)) R_{2j} V_j \\
 &\leq \varepsilon C U_j (1 + \alpha\varepsilon (\mathcal{W}_s + \beta U)) (\|R_{1j}\|_{L^2} + \|R_{2j}\|_{L^2}). \tag{2.14}
 \end{aligned}$$

Next, let us estimate the remainder terms  $R_{1j}$  and  $R_{2j}$ . In order to do so, we apply Proposition 4.6 and Proposition 4.1 from the Appendix in order to get:

$$\begin{aligned}
 \|R_{1j}\|_{L^2} &\leq C 2^{-js} c_j^2(t) \left\{ \beta \|\nabla V\|_{L^\infty} \|\nabla\eta\|_{B_{2,r}^{s-1}} + \beta \|\nabla\eta\|_{L^\infty} \|\nabla V\|_{B_{2,r}^{s-1}} + \right. \\
 &\quad \|\nabla W_1\|_{L^\infty} \|\nabla\eta\|_{B_{2,r}^{s-1}} + \|\nabla\eta\|_{L^{p_2}} \|\nabla W_1\|_{B_{p_1,r}^{s-1}} + \\
 &\quad \beta \|\nabla\eta\|_{L^\infty} \|\nabla V\|_{B_{2,r}^{s-1}} + \beta \|\nabla V\|_{L^\infty} \|\nabla\eta\|_{B_{2,r}^{s-1}} + \\
 &\quad \|\nabla h\|_{L^\infty} \|\nabla V\|_{B_{2,r}^{s-1}} + \|\nabla V\|_{L^{p_2}} \|\nabla h\|_{B_{p_1,r}^{s-1}} + \\
 &\quad \|\nabla h\|_{L^\infty} \|\nabla V\|_{B_{2,r}^{s-1}} + \|V\|_{L^{p_2}} \|\nabla h\|_{B_{p_1,r}^s} + \\
 &\quad \left. \|\operatorname{div} W_1\|_{L^\infty} \|\nabla\eta\|_{B_{2,r}^{s-1}} + \|\eta\|_{L^{p_2}} \|\operatorname{div} W_1\|_{B_{p_1,r}^s} \right\} \tag{2.15}
 \end{aligned}$$

and consequently:

$$\|R_{1j}\|_{L^2} \leq C 2^{-js} c_j^2(t) (\mathcal{W}_s U + U_s (\mathcal{W}_s + \beta U)).$$

Proceeding as above, we get a similar bound for  $R_{2j}$ . Thus, we get that:

$$T_5 \leq \varepsilon C 2^{-js} c_j^2(t) U_j (1 + \alpha\varepsilon (\mathcal{W}_s + \beta U)) (\mathcal{W}_s U + U_s (\mathcal{W}_s + \beta U)) \tag{2.16}$$

When  $\alpha = 0$ ,  $T_6 = T_7 = 0$  and we are in the position of obtaining the first estimate. Combining (2.10), (2.11), (2.12), (2.13) and (2.16) we get that:

$$\frac{d}{dt} U_j^2 \leq \sqrt{\varepsilon} C U_j^2 (\mathcal{W}_s + \beta U) + \varepsilon C 2^{-js} c_j^5 U_j (F_s + \mathcal{W}_s U + U_s (\mathcal{W}_s + \beta U)). \tag{2.17}$$

Time integration of (2.17) reveals the following:

$$\begin{aligned}
 U_j(t) &\leq U_j(0) + \sqrt{\varepsilon} C \int_0^t U_j(\tau) (\mathcal{W}_s(\tau) + \beta U(\tau)) d\tau + \\
 &\quad \sqrt{\varepsilon} C \int_0^t 2^{-j\sigma} c_j^5(\tau) (F_s(\tau) + \mathcal{W}_s(\tau) U(\tau) + U_s(\tau) (\mathcal{W}_s(\tau) + \beta U(\tau))) d\tau.
 \end{aligned}$$

Multiplying the last inequality with  $2^{js}$  and performing an  $\ell^r(\mathbb{Z})$ -summation we end up with:

$$\begin{aligned} U_s(t) &\leq U_s(0) + \sqrt{\varepsilon}C \int_0^t F_s(\tau) d\tau + \sqrt{\varepsilon}C \int_0^t \mathcal{W}_s(\tau) U(\tau) d\tau \\ &\quad + \sqrt{\varepsilon}C \int_0^t U_s(\tau) (\mathcal{W}_s(\tau) + \beta U(\tau)) d\tau. \\ &\leq U_s(0) + \sqrt{\varepsilon}C \int_0^t F_s(\tau) d\tau + \sqrt{\varepsilon}C \int_0^t U_s(\tau) (\mathcal{W}_s(\tau) + \beta U(\tau)) d\tau. \end{aligned} \quad (2.18)$$

Obviously, relation (2.18) and Gronwall's lemma imply the explosion criteria.

Let us now bound the term  $T_6$ :

$$\begin{aligned} T_6 &= -\frac{1}{2}\alpha\varepsilon \langle V_j, (\partial_t h + \beta \partial_t \eta) V_j \rangle_{L^2} - \frac{1}{2}\alpha\varepsilon^2 d \langle \nabla V_j, (\partial_t h + \beta \partial_t \eta) \nabla V_j \rangle_{L^2} \\ &\leq \alpha\varepsilon U_j^2 (\|\partial_t h\|_{L^\infty} + \beta \|\partial_t \eta\|_{L^\infty}). \end{aligned}$$

Using the first equation of  $(\mathcal{S}_\varepsilon(\mathcal{D}))$  we write:

$$\partial_t \eta = \varepsilon (I - \varepsilon b \Delta)^{-1} f - (I - \varepsilon b \Delta)^{-1} \operatorname{div} (V + \varepsilon (\eta W_1 + hV + \beta \eta V))$$

and using again relation (2.1) combined with the fact that  $(I - \varepsilon b \Delta)^{-1}$  has at most norm 1 when regarded as an  $L^2$  to  $L^2$  operator, we obtain that:

$$\begin{aligned} \|\partial_t \eta\|_{L^\infty} &\leq \left\| \varepsilon (I - \varepsilon b \Delta)^{-1} f - (I - \varepsilon b \Delta)^{-1} \operatorname{div} (V + \varepsilon (\eta W_1 + hV + \beta \eta V)) \right\|_{B_{2,1}^{\frac{2}{3}}} \\ &\leq \varepsilon \|f\|_{B_{2,r}^s} + \|V\|_{B_{2,r}^s} + \varepsilon \|\eta W_1 + hV + \beta \eta V\|_{B_{2,r}^s} \\ &\leq \varepsilon \|f\|_{B_{2,r}^s} + \|V\|_{B_{2,r}^s} + \varepsilon \beta \|(\eta, V)\|_{B_{2,r}^s}^2 + \varepsilon \|(\eta, V)\|_{B_{2,r}^s} \|(h, W_1)\|_{L^\infty} \\ &\quad + \varepsilon \|(\eta, V)\|_{L^{p_2}} \|(\nabla h, \nabla W_1)\|_{B_{p_1,r}^{s-1}} \\ &\leq C(U_s + \varepsilon F_s + \varepsilon U_s (\mathcal{W}_s + \beta U_s)). \end{aligned} \quad (2.19)$$

Thus, putting together the last estimates, we find that:

$$T_6 \leq \alpha\varepsilon U_j^2 \left( \mathcal{W}_s + \beta U_s + \varepsilon \beta F_s + \varepsilon (\mathcal{W}_s + \beta U_s)^2 \right). \quad (2.20)$$

Finally, let us estimate the last term:

$$T_7 = -\alpha d \varepsilon^2 \langle \partial_t \nabla V_j, \nabla (\beta \eta + h) \otimes V_j \rangle_{L^2} \leq \alpha d \varepsilon^2 \|\partial_t \nabla V_j\|_{L^2} \|V_j\|_{L^2} (\beta \|\nabla \eta\|_{L^\infty} + \|\nabla h\|_{L^\infty}).$$

Using the second equation of  $(\mathcal{S}_\varepsilon(\mathcal{D}))$  we write that:

$$\begin{aligned} \partial_t \nabla V_j &= \varepsilon (I - \varepsilon d \Delta)^{-1} \nabla (g_j) - (I - \varepsilon d \Delta)^{-1} \nabla^2 \eta_j - \varepsilon (I - \varepsilon d \Delta)^{-1} \nabla \Delta_j [(W_2 + \beta V) \cdot \nabla V] \\ &\quad - \varepsilon (I - \varepsilon d \Delta)^{-1} \nabla \Delta_j (V \cdot \nabla W_3) \end{aligned}$$

and because  $(\varepsilon d)^{\frac{1}{2}} (I - \varepsilon d \Delta)^{-1} \nabla$  and  $\varepsilon d (I - \varepsilon d \Delta)^{-1} \nabla^2$  have  $O(1)$ -norms when regarded as operators from  $L^2$  to  $L^2$ , we can write that

$$\begin{aligned} \varepsilon d \|\partial_t \nabla V_j\|_{L^2} &\leq C \left( \varepsilon^{\frac{3}{2}} \sqrt{d} \|g_j\|_{L^2} + \|\eta_j\|_{L^2} + \varepsilon^{\frac{3}{2}} \sqrt{d} \|\Delta_j [(W_2 + \beta V) \cdot \nabla V]\|_{L^2} + \varepsilon^{\frac{3}{2}} \sqrt{d} \|\Delta_j (V \cdot \nabla W_3)\|_{L^2} \right) \\ &\leq C \max \left\{ 1, \sqrt{\varepsilon d} \right\} \left( U_j + 2^{-js} c_j^4 \left( \varepsilon F_s + \varepsilon (\mathcal{W}_s + \beta U_s)^2 \right) \right). \end{aligned}$$

Thus, we get that:

$$T_7 \leq \alpha\varepsilon U_j^2 (\mathcal{W}_s + \beta U) + \alpha\varepsilon C 2^{-js} c_j^4(t) U_j \left( \varepsilon F_s (\mathcal{W}_s + \beta U) + \varepsilon U_s (\mathcal{W}_s + \beta U_s)^2 \right). \quad (2.21)$$

Let us consider  $\alpha = 1$ . Putting together estimates (2.10)-(2.16) we get that:

$$\begin{aligned} \frac{d}{dt} \left[ \int \eta_j^2 + \varepsilon b |\nabla \eta_j|^2 + (1 + \varepsilon (\eta + h)) (V_j^2 + \varepsilon d \nabla V_j : \nabla V_j) \right] &\leq \varepsilon C U_j^2 \left( \varepsilon \beta F_s + \mathcal{W}_s + \beta U_s + \varepsilon (\mathcal{W}_s + \beta U_s)^2 \right) \\ &\quad + \varepsilon C 2^{-js} c_j(t) U_j (F_s (1 + \varepsilon (\mathcal{W}_s + \beta U_s)) + U_s (\mathcal{W}_s + \beta U_s) (1 + \varepsilon (\mathcal{W}_s + \beta U_s))). \end{aligned} \quad (2.22)$$

### 2.1.2 Existence and uniqueness of solutions

We are now in the position to prove the existence and uniqueness of solutions for system  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . We will use the so called Friedrichs method. For all  $m \in \mathbb{N}$ , let us consider  $\mathbb{E}_m$  the low frequency cut-off operator defined by:

$$\mathbb{E}_m f = \mathcal{F}^{-1} \left( \chi_{B(0,m)} \hat{f} \right).$$

We define the space

$$L_m^2 = \left\{ f \in L^2 : \text{Supp } \hat{f} \subset B(0, m) \right\}$$

which, endowed with the  $\|\cdot\|_{L^2}$ -norm is a Banach space. Let us observe that due to Bernstein's lemma, all Sobolev norms are equivalent on  $L_m^2$ . For all  $m \in \mathbb{N}$ , we consider the following differential equation on  $L_m^2$ :

$$\begin{cases} \partial_t \eta = F_m(\eta, V), \\ \partial_t V = G_m(\eta, V), \\ \eta|_{t=0} = \mathbb{E}_m \eta_0, \quad V|_{t=0} = \mathbb{E}_m V_0, \end{cases} \quad (2.23)$$

where  $(F_m, G_m) : L_m^2 \times (L_m^2)^n \rightarrow L_m^2 \times (L_m^2)^n$  are defined by:

$$F_m(\eta, V) = -\mathbb{E}_m \left( (I - \varepsilon b \Delta)^{-1} [\text{div } V + \varepsilon \text{div}(\eta W_1 + hV + \beta \eta V) - \varepsilon f] \right), \quad (2.24)$$

$$G_m(\eta, V) = -\mathbb{E}_m \left( (I - \varepsilon d \Delta)^{-1} [\nabla \eta + \varepsilon (W_2 + \beta) \cdot \nabla V + \varepsilon V \cdot \nabla W_3 - \varepsilon g] \right). \quad (2.25)$$

It transpires that due to the equivalence of the Sobolev norm,  $(F_m, G_m)$  is continuous and locally Lipschitz on  $L_m^2 \times (L_m^2)^n$ . Thus, the classical Picard theorem ensures that there exists a nonnegative time  $T_m > 0$  and a unique solution  $(\eta^m, V^m) : \mathcal{C}^1([0, T_m], L_m^2 \times (L_m^2)^n)$ . Let us denote by

$$\begin{cases} U_j^m(t) = \left( \|(\Delta_j \eta^m(t), \Delta_j V^m(t))\|_{L^2}^2 + \varepsilon \left\| \left( \sqrt{b} \nabla \Delta_j \eta^m(t), \sqrt{d} \nabla \Delta_j V^m(t) \right) \right\|_{L^2}^2 \right)^{\frac{1}{2}}, \\ U_s^m(t) = \left\| (2^{js} U_j^m(t))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}, \quad U^m(t) = \|(\eta^m, \nabla \eta^m, V^m, \nabla V^m)\|_{L^\infty \cap L^{p_2}}. \end{cases}$$

Thanks to the property  $\mathbb{E}_m^2 = \mathbb{E}_m$ , we get that the estimate obtained in (2.18) still holds true for  $(\eta^m, V^m)$ , namely:

$$\begin{aligned} U_s^m(t) &\leq U_s^m(0) + \sqrt{\varepsilon} C \int_0^t F_s(\tau) d\tau + \sqrt{\varepsilon} C \int_0^t U_s^m(\tau) (U^m(\tau) + \mathcal{W}_s(\tau)) d\tau \\ &\leq U_s(0) + \sqrt{\varepsilon} C \int_0^t (F_s(\tau) + \mathcal{W}_s^2(\tau)) d\tau + \sqrt{\varepsilon} C \int_0^t (U_s^m(\tau))^2 d\tau. \end{aligned}$$

We consider

$$T' = \sup \left\{ T > 0 : \sqrt{\varepsilon} C \int_0^t (F_s(\tau) + \mathcal{W}^2(\tau)) d\tau \leq U_s(0) \right\}$$

and

$$\bar{T}_m = \sup \{ T > 0 : U_s(\tau) \leq 3U_s(0) \}.$$

Gronwall's lemma ensures the existence of a constant  $\bar{C} = \bar{C}(s, b, d)$  such that:

$$\bar{T}_m \geq \min \left\{ \frac{\bar{C}}{\sqrt{\varepsilon} U_s(0)}, T' \right\} := \bar{T}$$

and thus, the sequence  $(\eta^m, V^m) \in \mathcal{C}([0, T], B_{2,r}^{s_1} \times (B_{2,r}^{s_2})^n)$  is uniformly bounded i.e.

$$\|(\eta^m, V^m)\|_E \leq 3U_s(0). \quad (2.26)$$

From the relation

$$\partial_t \eta^m = -\mathbb{E}_m \left( (I - \varepsilon b \Delta)^{-1} [\text{div } V^m + \varepsilon \text{div}(\eta^m W_1 + hV^m + \beta \eta^m V^m) - \varepsilon f] \right)$$

and from (2.26) we get that the sequence  $(\partial_t \eta^m)_{m \in \mathbb{N}}$  is uniformly bounded in  $B_{2,r}^{s-1}$  on  $[0, \bar{T}]$ . Next, considering for all  $p \in \mathbb{N}$  a smooth function  $\phi_p$  such that:

$$\begin{cases} \text{Supp } \phi_p \subset B(0, p+1), \\ \phi_p = 1 \text{ on } B(0, p) \end{cases}$$

it follows that for each  $p \in \mathbb{N}$ , the sequence  $(\phi_p \eta^m)_{m \in \mathbb{N}}$  is uniformly equicontinuous on  $[0, \bar{T}]$  and that for all  $t \in [0, \bar{T}]$ , the set  $\{\phi_p \eta^m(t) : m \in \mathbb{N}\}$  is relatively compact in  $B_{2,r}^{s-1}$ . Thus, the Ascoli-Arzelà Theorem combined with Proposition 4.5 and with Cantor's diagonal process provides us a subsequence of  $(\eta^m)_{m \in \mathbb{N}}$  and a tempered distribution  $\eta \in \mathcal{C}([0, T], S')$  such that for all  $\phi \in \mathcal{D}(\mathbb{R}^n)$ :

$$\phi \eta^m \rightarrow \phi \eta \text{ in } \mathcal{C}([0, T], B_{2,r}^{s-1}).$$

Moreover, owing to the Fatou property for Besov spaces, see Proposition 4.3, we get that  $\eta \in L^\infty([0, T], B_{2,r}^{s_1})$  and thus, using interpolation we get that:

$$\phi \eta^m \rightarrow \phi \eta \text{ in } \mathcal{C}([0, T], B_{2,r}^{s_b - \gamma}) \quad (2.27)$$

for any  $\gamma > 0$ . Of course, using the same argument we can construct a  $V \in L^\infty([0, \bar{T}], (B_{2,r}^{s_2})^n)$  such that for any  $\psi \in (\mathcal{D}(\mathbb{R}^n))^n$ :

$$\psi V^m \rightarrow \psi V \text{ in } \mathcal{C}([0, T], (B_{2,r}^{s_d - \gamma})^n) \quad (2.28)$$

for any  $\gamma > 0$ . Also, by the Fatou property in Besov spaces we get that  $(\eta, V) \in L^\infty([0, \bar{T}], B_{2,r}^{s_b} \times (B_{2,r}^{s_d})^n)$ . We claim that the properties enlisted above allow us to pass to the limit when  $m \rightarrow \infty$  in the equation verified by  $\eta^m$  and  $V^m$ . Let us show on two examples, how this process is carried out in practice. Let  $\phi \in \mathcal{D}(\mathbb{R}^n)$  and let us write that:

$$\begin{aligned} \left| \int \phi \operatorname{div} [(\eta^m - \eta) W_1] \right| &= \left| \int (\eta^m - \eta) W_1 \nabla \phi \right| \leq \left| \int (\eta^m - \eta) (S_q W_1) \nabla \phi \right| + \left| \int (\eta^m - \eta) ((Id - S_q) W_1) \nabla \phi \right| \\ &\leq \left| \int (\eta^m - \eta) (S_q W_1) \nabla \phi \right| + C \| (Id - S_q) W_1 \|_{B_{p_1, r}^{s+1}} \| (\eta^m - \eta) \nabla \phi \|_{B_{p'_1, r'}^{s-1}} \\ &\leq \left| \int (\eta^m - \eta) (S_q W_1) \nabla \phi \right| + C \| (Id - S_q) \nabla W_1 \|_{B_{p_1, r}^s} \| (\eta^m - \eta) \nabla \phi \|_{B_{p'_1, \infty}^0} \\ &\leq \left| \int (\eta^m - \eta) (S_q W_1) \nabla \phi \right| + C \| (Id - S_q) \nabla W_1 \|_{B_{p_1, r}^s} (\| \eta^m \|_{L^{p_2}} + \| \eta \|_{L^{p_2}}) \| \nabla \phi \|_{L^2} \\ &\leq \left| \int (\eta^m - \eta) (S_q W_1) \nabla \phi \right| + C U_s(0) \| (Id - S_q) \nabla W_1 \|_{B_{p_1, r}^s} \| \nabla \phi \|_{L^2}. \end{aligned} \quad (2.29)$$

The fact that the first term of (2.29) tends to zero as  $m \rightarrow \infty$  is a consequence of (2.27). The second term tends to zero as  $q \rightarrow \infty$  owing to the fact that  $\nabla W_1 \in B_{p_1, r}^s$ . Let us also show how to deal with the nonlinear terms. We write that

$$\begin{aligned} \left| \int \phi \operatorname{div} (\eta^m V^m - \eta V) \right| &= \left| \int (\eta^m V^m - \eta V) \nabla \phi \right| \leq \left| \int \eta^m (V^m - V) \nabla \phi \right| + \left| \int (\eta^m - \eta) V \nabla \phi \right| \\ &\leq C \| V^m \nabla \phi - V \nabla \phi \|_{B_{2,r}^{s-1}} \| \eta^m \|_{B_{2,r'}^{1-s}} + \left| \int (\eta^m - \eta) V \nabla \phi \right| \\ &\leq C \| V^m \nabla \phi - V \nabla \phi \|_{B_{2,r}^{s-1}} \| \eta^m \|_{B_{2,r}^s} + \left| \int (\eta^m - \eta) V \nabla \phi \right| \\ &\leq C U_s(0) \| V^m \nabla \phi - V \nabla \phi \|_{B_{2,r}^{s-1}} + \left| \int (\eta^m - \eta) V \nabla \phi \right|. \end{aligned} \quad (2.30)$$

The first term of (2.30) tends to zero as  $m \rightarrow \infty$ , owing to relation (2.28). In order to show that the second term of (2.30) tends to zero as  $m \rightarrow \infty$  one proceeds exactly like we did in (2.29).

Recovering the time regularity of  $(\eta, V)$  is again classical. One can show for example that for all  $j \in \mathbb{Z}$  we have

$$S_j \eta \in \mathcal{C}([0, T], B_{2,r}^{s_1})$$

and by using energy estimates:

$$\lim_{j \rightarrow \infty} \|\eta - S_j \eta\|_{L_T^\infty(B_{2,r}^{s_1})} = 0.$$

As for the uniqueness of solutions let us consider  $(\eta^1, V^1)$ ,  $(\eta^2, V^2)$  two solutions of  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . The system verified by

$$(\delta\eta, \delta V) = (\eta^1 - \eta^2, V^1 - V^2)$$

is the following:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \delta\eta + \operatorname{div} \delta V + \varepsilon \operatorname{div} (\delta\eta \tilde{W}_1 + \tilde{h} \delta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t \delta V + \nabla \delta\eta + \varepsilon \tilde{W}_2 \cdot \nabla \delta V + \delta V \cdot \nabla \tilde{W}_3 = 0 \\ \eta|_{t=0} = 0, \quad V|_{t=0} = 0, \end{cases} \quad (2.31)$$

with

$$\begin{cases} \tilde{h} = h + \beta \eta^2, & \tilde{W}_1 = W_1 + \beta V^1, \\ \tilde{W}_2 = W_2 + \beta V^1, & \tilde{W}_3 = W_3 + \beta V^2. \end{cases}$$

Let us multiply the first equation of (2.31) with  $\delta\eta$  and the second one with  $\delta V$  such that by repeated integration by parts, one obtains:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \delta U^2(t) &\leq C \left( \varepsilon \|\operatorname{div} \tilde{W}_1\|_{L^\infty} + \varepsilon \|\operatorname{div} \tilde{W}_2\|_{L^\infty} \right) \delta U^2(t) \\ &\quad + C \left( \varepsilon \|\nabla \tilde{W}_3\|_{L^\infty} + \frac{\sqrt{\varepsilon} (\|\tilde{h}\|_{L^\infty} + \|\nabla \tilde{h}\|_{L^\infty})}{\max\{\sqrt{b}, \sqrt{d}\}} \right) \delta U^2(t) \end{aligned} \quad (2.32)$$

with

$$\delta U^2(t) \stackrel{not.}{=} \int \left( \|\delta\eta, \delta V\|_{L^2}^2 + \|(b \nabla \delta\eta, d \nabla \delta V)\|_{L^2}^2 \right).$$

As  $\delta U(0) = 0$ , uniqueness follows by Gronwall's Lemma and thus the proof of Theorem 2.1 is achieved.

## 2.2 The lower bound on the time of existence

In this section we prove Theorem 2.2. Let us consider  $(\eta_0, V_0) \in X_{b,d,r}^{s,\varepsilon}$  and let us denote by

$$R_0^\varepsilon \stackrel{not.}{=} \|(\eta_0, V_0)\|_{X_{b,d,r}^{s,\varepsilon}}.$$

Then, according to Theorem 2.1, for all  $\varepsilon > 0$ , there exists a unique maximal solution of  $\mathcal{S}_\varepsilon(\mathcal{D}^\varepsilon)$  which we denote by  $(\eta^\varepsilon(t), V^\varepsilon(t)) \in X_{b,d,r}^{s,\varepsilon}$ . We introduce the following notations:

$$\begin{cases} U_j^2(\varepsilon, t) = \|(\eta_j^\varepsilon(t), V_j^\varepsilon(t))\|_{L^2}^2 + \varepsilon \left\| \left( \sqrt{b} \nabla \eta_j^\varepsilon(t), \sqrt{d} \nabla V_j^\varepsilon(t) \right) \right\|_{L^2}^2, \\ \|(\eta^\varepsilon(t), V^\varepsilon(t))\|_{X_{b,d,r}^{s,\varepsilon}} = U_s(\varepsilon, t) = \left\| (2^{js} U_j(\varepsilon, t))_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})}. \end{cases} \quad (2.33)$$

Let us consider

$$T_\star^\varepsilon = \sup \left\{ T \in [0, T^\varepsilon] : \sup_{t \in [0, T]} \|(\eta^\varepsilon(t), V^\varepsilon(t))\|_{X_{b,d,r}^{s,\varepsilon}} \leq (1 + e\sqrt{\varepsilon}) R_0^\varepsilon \right\},$$

and

$$\varepsilon_0 = \min \{\varepsilon_{01}, \varepsilon_{02}, \varepsilon_{03}, \varepsilon_{04}\},$$

where

$$\begin{cases} \varepsilon_{01} = \frac{3}{4C_1(1+e\sqrt{\varepsilon})R_0^1+4} \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \|h^{\tilde{\varepsilon}}(\tau)\|_{L^\infty}, & \varepsilon_{02} = \frac{1}{2(1+e\sqrt{\varepsilon})R_0^1+2} \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau), \\ \varepsilon_{03} = \frac{(1+e\sqrt{\varepsilon})R_0^0 + \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau)}{2 \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau)}, & \varepsilon_{04} = \frac{1}{2(1+e\sqrt{\varepsilon})R_0^1+2} \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau). \end{cases}$$

where  $C_1$  is the constant appearing in the embedding  $B_{2,1}^{\frac{n}{2}} \hookrightarrow L^\infty$  i.e.

$$\|f\|_{L^\infty} \leq C_1 \|f\|_{B_{2,1}^{\frac{n}{2}}}.$$

Let us put

$$\tilde{C} = \min \left\{ \frac{R_0^0}{3eC \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0,T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau)}, \frac{1}{16C(1+e\sqrt{7})R_0^1}, \frac{1}{16C \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0,T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau)} \right\} \quad (2.34)$$

where  $C$  is the universal constant appearing in (2.22). We claim that for all  $\varepsilon \leq \varepsilon_0$

$$T_\star^\varepsilon \geq \min \left\{ \frac{\tilde{C}}{\varepsilon}, T^\varepsilon \right\}.$$

Let us suppose that this is not the case. Then there exists an  $\varepsilon > 0$  such that

$$T_\star^\varepsilon < \min \left\{ \frac{\tilde{C}}{\varepsilon}, T^\varepsilon \right\}. \quad (2.35)$$

We begin by writing that

$$\frac{1}{4} \leq 1 + \varepsilon (\eta^\varepsilon(t, x) + h^\varepsilon(t, x)) \leq \frac{7}{4},$$

for all  $(t, x) \in [0, T_\star^\varepsilon] \times \mathbb{R}^n$ , owing to the fact that

$$\varepsilon \leq \varepsilon_{01}. \quad (2.36)$$

Thus, denoting by

$$N_j^2(\varepsilon, t) := \int_{\mathbb{R}^n} (\eta_j^\varepsilon)^2(t) + \varepsilon b |\nabla \eta_j^\varepsilon(t)|^2 + (1 + \varepsilon (\eta^\varepsilon(t) + h^\varepsilon(t))) \left( |V_j^\varepsilon|^2(t) + \varepsilon d \nabla V_j^\varepsilon(t) : \nabla V_j^\varepsilon(t) \right)$$

we see that for all  $t \in [0, T_\star^\varepsilon]$  we get that:

$$\frac{1}{2} U_j(\varepsilon, t) \leq N_j(\varepsilon, t) \leq \frac{\sqrt{7}}{2} U_j(\varepsilon, t). \quad (2.37)$$

Recall that according to (2.22) we have that:

$$\frac{d}{dt} N_j^2(\varepsilon, t) \leq \varepsilon C U_j^2(\varepsilon, t) \left( \varepsilon F_s^\varepsilon(t) + \mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon, t) + \varepsilon (\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon, t))^2 \right) \quad (2.38)$$

$$+ \varepsilon C 2^{-js} c_j^\varepsilon(t) U_j(\varepsilon, t) F_s^\varepsilon(t) \{1 + \varepsilon (\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon, t))\} \quad (2.39)$$

$$+ \varepsilon C 2^{-js} c_j^\varepsilon(t) U_j(\varepsilon, t) U_s(\varepsilon, t) (\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon, t)) \{1 + \varepsilon (\mathcal{W}_s^\varepsilon(t) + U_s(\varepsilon, t))\} \quad (2.40)$$

and using (2.37) we get that:

$$\begin{aligned} U_j(\varepsilon, t) &\leq \sqrt{7} U_j(\varepsilon, 0) + 2\varepsilon C \int_0^t U_j(\varepsilon, \tau) \left( \varepsilon F_s^\varepsilon(\tau) + \mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau) + \varepsilon (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau))^2 \right) \\ &+ 2\varepsilon C 2^{-js} \int_0^t c_j(F_s^\varepsilon(\tau) (1 + \varepsilon (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau))) + U_s(\varepsilon, \tau) (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau)) (1 + \varepsilon (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau)))) d\tau. \end{aligned} \quad (2.41)$$

Multiplying (2.41) with  $2^{js}$  and performing an  $\ell^r(\mathbb{Z})$ -summation we end up with

$$U_s(\varepsilon, t) \leq \sqrt{7} U_s(\varepsilon, 0) + 2\varepsilon C \int_0^t F_s^\varepsilon(\tau) (1 + \varepsilon (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau))) d\tau$$

$$\begin{aligned}
 & + 4\varepsilon C \int_0^t U_s(\varepsilon, \tau) \left( \varepsilon F_s^\varepsilon(\tau) + \mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau) + \varepsilon (\mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau))^2 \right) d\tau \\
 & \leq \sqrt{7} R_0^\varepsilon + 3\varepsilon C T_\star^\varepsilon \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau) \\
 & \quad + 4\varepsilon C \int_0^t U_s(\varepsilon, \tau) \left( \varepsilon F_s^\varepsilon(\tau) + \mathcal{W}_s^\varepsilon(\tau) + U_s(\varepsilon, \tau) + \varepsilon (\mathcal{W}_s^\varepsilon + U_s(\varepsilon, \tau))^2 \right) d\tau, \tag{2.42}
 \end{aligned}$$

owing to the fact that  $\varepsilon$  has been chosen such that:

$$\varepsilon \leq \varepsilon_{02}. \tag{2.43}$$

Using the definition of  $\tilde{C}$  we get that

$$T_\star^\varepsilon < \frac{R_0^0}{3\varepsilon e C \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau)} < \frac{R_0^\varepsilon}{3\varepsilon e C \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} F_s^{\tilde{\varepsilon}}(\tau)} \tag{2.44}$$

and consequently, we get that:

$$U_s(\varepsilon, t) \leq \left( e^{-1} + \sqrt{7} \right) R_0^\varepsilon + 8\varepsilon C \left( \left( 1 + e\sqrt{7} \right) R_0^\varepsilon + \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau) \right) \int_0^t U_s(\varepsilon, \tau) d\tau, \tag{2.45}$$

where we have used that  $\varepsilon \leq \varepsilon_{03}$  respectively  $\varepsilon \leq \varepsilon_{04}$ . The estimate (2.45) along with Gronwall's Lemma imply that:

$$U_s(\varepsilon, t) \leq \left( e^{-1} + \sqrt{7} \right) R_0^\varepsilon \exp \left( 8\varepsilon C T_\star^\varepsilon \left( \left( 1 + e\sqrt{7} \right) R_0^\varepsilon + \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau) \right) \right).$$

Owing to

$$\begin{aligned}
 T_\star^\varepsilon & < \frac{1}{\varepsilon} \min \left\{ \frac{1}{16C (1 + e\sqrt{7}) R_0^1}, \frac{1}{16C \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau)} \right\} \\
 & \leq \frac{1}{8\varepsilon C \left( (1 + e\sqrt{7}) R_0^1 + \sup_{\tilde{\varepsilon} \in [0,1]} \sup_{\tau \in [0, T^{\tilde{\varepsilon}}]} \mathcal{W}_s^{\tilde{\varepsilon}}(\tau) \right)}
 \end{aligned}$$

we obtain that

$$\sup_{t \in [0, T_\star^\varepsilon]} U_s(\varepsilon, t) < \left( 1 + e\sqrt{7} \right) R_0^\varepsilon$$

which in view of the time continuity of  $(\eta^\varepsilon, V^\varepsilon)$  is a contradiction. Thus, our initial suppositions is false. Thus, if  $\varepsilon \leq \varepsilon_0$  we get that

$$T_\star^\varepsilon \geq \min \left\{ T^\varepsilon, \frac{\tilde{C}}{\varepsilon} \right\}.$$

By the definition of  $T_\star^\varepsilon$  we have that

$$\sup_{t \in [0, T_\star^\varepsilon]} \|(\eta^\varepsilon(t), V^\varepsilon(t))\|_{X_{b,d,r}^{s,\varepsilon}} \leq \left( 1 + e\sqrt{7} \right) R_0^\varepsilon. \tag{2.46}$$

Observe that according to the uniform bounds of (2.46) and relation (2.19) we get that

$$\sup_{t \in [0, T_\star^\varepsilon]} \|\partial_t \eta^\varepsilon(t)\|_{L^\infty} \leq 2C' \left( 1 + e\sqrt{7} \right) R_0^\varepsilon,$$

where  $C'$  is the constant appearing in relation (2.19). This ends the proof of Theorem 2.2.



### 3 Proofs of the main results

#### 3.1 The 1d case: proof of Theorem 1.1

In the following lines we prove the 1-dimensional result announced in Theorem 1.1. For the reader's convenience, let us write below the system:

$$\begin{cases} (I - \varepsilon b \partial_{xx}^2) \partial_t \bar{\eta} + \partial_x \bar{u} + \varepsilon \partial_x (\bar{\eta} \bar{u}) = 0, \\ (I - \varepsilon d \partial_{xx}^2) \partial_t \bar{u} + \partial_x \bar{\eta} + \varepsilon \bar{u} \partial_x \bar{u} = 0, \\ \bar{\eta}|_{t=0} = \eta_0, \quad \bar{u}|_{t=0} = u_0. \end{cases} \quad (3.1)$$

The strategy of the proof is the following. First, we split the initial data into low-high frequency parts i.e. :

$$\begin{aligned} (\eta_0, u_0) &= (\Delta_{-1} \eta_0, \Delta_{-1} u_0) + ((I - \Delta_{-1}) \eta_0, (I - \Delta_{-1}) u_0) \\ &\stackrel{not.}{=} (\eta_0^{low}, u_0^{low}) + (\eta_0^{high}, u_0^{high}). \end{aligned}$$

We solve the linear acoustic waves system with initial data  $(\eta_0^{low}, u_0^{low})$ . Searching for a solution  $(\bar{\eta}, \bar{u})$  of (3.1) in the form  $(\eta + \eta_L, u + u_L)$  we observe that in fact  $(\eta, u)$  verifies a system of type  $(\mathcal{S}_\varepsilon(\mathcal{D}))$ . In view of the uniform bounds (with respect to time) of  $(\eta_L, u_L)$  we obtain the existence of the pair  $(\eta, u)$  as a consequence of Corollary 3. Finally, we prove the uniqueness property by performing classical  $L^2$ -energy estimates with respect to the system of equations governing the difference of two solutions with the same initial data.

Owing to Bernstein's Lemma (4.1) we have:

$$\|(\eta_0^{low}, u_0^{low})\|_{L^\infty} \leq C_1 \|(\eta_0, u_0)\|_{L^\infty}. \quad (3.2)$$

Moreover, it transpires that  $(\eta_0^{high}, u_0^{high}) \in X_{b,d,r}^s$  and that

$$\left\| (\eta_0^{high}, u_0^{high}) \right\|_{X_{b,d,r}^{s,\varepsilon}} \leq C_2 \|(\partial_x \eta_0, \partial_x u_0)\|_{X_{b,d,r}^{s-1,\varepsilon}}. \quad (3.3)$$

Let us consider  $(\eta_L, u_L) \in \mathcal{C}([0, \infty), E_{b,d,r}^s(\mathbb{R}))$  the unique solution of the linear acoustic waves system:

$$\begin{cases} \partial_t \eta_L + \partial_x u_L = 0, \\ \partial_t u_L + \partial_x \eta_L = 0, \\ \eta_L|_{t=0} = \eta_0^{low}, \quad u_L|_{t=0} = u_0^{low} \end{cases} \quad (3.4)$$

which is given explicitly by the following relation:

$$\begin{cases} 2\eta_L(t, x) = (\eta_0^{low}(x+t) + \eta_0^{low}(x-t)) + (u_0^{low}(x+t) - u_0^{low}(x-t)), \\ 2u_L(t, x) = (\eta_0^{low}(x+t) - \eta_0^{low}(x-t)) + (u_0^{low}(x+t) + u_0^{low}(x-t)), \end{cases} \quad (3.5)$$

for all  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ . From (3.5) we conclude that:

$$\begin{aligned} \|(\eta_L(t), u_L(t))\|_{L^\infty} &\leq 2C_1 \|(\eta_0, u_0)\|_{L^\infty}, \\ \|(\partial_x \eta_L(t), \partial_x u_L(t))\|_{B_{2,r}^\sigma} &\leq 2C_1 \|(\partial_x \eta_0, \partial_x u_0)\|_{B_{2,\infty}^0} \end{aligned} \quad (3.6)$$

for all  $\sigma \geq 0$ . In particular, we also have

$$\|(\eta_L(t), u_L(t))\|_{E_{b,d,r}^{s,\varepsilon}} \leq 2C_1 \|(\eta_0, u_0)\|_{E_{b,d}^{s,\varepsilon}} \quad (3.7)$$

for all  $t \geq 0$ . Let us consider:

$$\begin{cases} f_L = -\eta_L \partial_x u_L - u_L \partial_x \eta_L - b \partial_{xxx}^3 u_L, \\ g_L = -u_L \partial_x u_L - d \partial_{xxx}^3 \eta_L. \end{cases}$$

Owing to the fact that  $(\widehat{\eta_L}, \widehat{u_L})$  is supported in a ball centered at the origin, we obtain that

$$\|(f_L(t), g_L(t))\|_{B_{2,r}^s} \leq C_s \left\{ \|(\eta_L, u_L)\|_{L^\infty} \|(\partial_x \eta_L, \partial_x u_L)\|_{B_{2,r}^s} + \|(\partial_x \eta_L, \partial_x u_L)\|_{L^\infty} \|(\partial_x \eta_L, \partial_x u_L)\|_{B_{2,r}^{s-1}} \right\}$$

$$\begin{aligned}
 & + b \left\| \partial_{xxx}^3 u_L \right\|_{B_{2,r}^s} + d \left\| \partial_{xxx}^3 \eta_L \right\|_{B_{2,r}^s} \Big\} \\
 & \leq C_s \left\| (\partial_x \eta_0, \partial_x u_0) \right\|_{B_{2,\infty}^0} \left( \left\| (\eta_0, u_0) \right\|_{L^\infty} + \left\| (\partial_x \eta_0, \partial_x u_0) \right\|_{B_{2,\infty}^0} + b + d \right). \tag{3.8}
 \end{aligned}$$

Owing to the uniform bounds of  $(\eta_L, u_L, f_L, g_L)$  announced in (3.6) and in (3.8) we can apply Corollary 3 with  $p_1 = 2$ ,  $p_2 = \infty$  in order to obtain the existence of two positive reals  $\varepsilon_0, C$  depending on  $s, b, d$  and the norm of the initial data such that for any  $\varepsilon \leq \varepsilon_0$  we may consider  $(\eta^\varepsilon, u^\varepsilon) \in \mathcal{C} \left( [0, \frac{C}{\varepsilon}], X_{b,d,r}^s \right)$  the solution of

$$\begin{cases} (I - \varepsilon b \partial_{xx}^2) \partial_t \eta + \partial_x u + \varepsilon \partial_x (\eta u_L + \eta_L u + \eta u) = \varepsilon f_L, \\ (I - \varepsilon d \partial_{xx}^2) \partial_t u + \partial_x \eta + \varepsilon (u + u_L) \partial_x u + u \partial_x u_L = \varepsilon g_L, \\ \eta|_{t=0} = \eta_0^{high}, \quad u|_{t=0} = u_0^{high} \end{cases} \tag{3.9}$$

which satisfies

$$\begin{aligned}
 \sup_{t \in [0, \frac{C}{\varepsilon}]} \left( \left\| (\eta^\varepsilon(t), u^\varepsilon(t)) \right\|_{X_{b,d,r}^{s,\varepsilon}} + \left\| \partial_t \eta^\varepsilon(t) \right\|_{L^\infty} \right) & \leq \tilde{C} \left\| (\eta_0^{high}, u_0^{high}) \right\|_{X_{b,d,r}^{s,\varepsilon}} \\
 & \leq \tilde{C} C_2 \left\| (\partial_x \eta_0, \partial_x u_0) \right\|_{X_{b,d,r}^{s-1,\varepsilon}}, \tag{3.10}
 \end{aligned}$$

for some numerical constant  $\tilde{C}$ . Considering

$$\bar{\eta}^\varepsilon = \eta^\varepsilon + \eta_L, \quad \bar{u}^\varepsilon = u^\varepsilon + u_L$$

we see that  $(\bar{\eta}^\varepsilon, \bar{u}^\varepsilon) \in \mathcal{C} \left( [0, \infty), E_{b,d,r}^s(\mathbb{R}) \right)$  and for all  $t \in [0, \frac{C}{\varepsilon}]$  we have that:

$$\begin{aligned}
 & \left\| (\bar{\eta}^\varepsilon(t), \bar{u}^\varepsilon(t)) \right\|_{E_{b,d}^{s,\varepsilon}} + \left\| \partial_t \bar{\eta}^\varepsilon(t) \right\|_{L^\infty} \\
 & \leq \left\| (\eta^\varepsilon(t), u^\varepsilon(t)) \right\|_{E_{b,d,r}^{s,\varepsilon}} + \left\| \partial_t \eta^\varepsilon(t) \right\|_{L^\infty} + \left\| (\eta_L(t), u_L(t)) \right\|_{E_{b,d,r}^{s,\varepsilon}} + \left\| \partial_t \eta_L(t) \right\|_{L^\infty} \\
 & \leq 2C_1 \left\| (\eta_0, u_0) \right\|_{E_{b,d,r}^{s,\varepsilon}} + \left\| \partial_x u_L(t) \right\|_{L^\infty} + \tilde{C} C_2 \left\| (\partial_x \eta_0, \partial_x u_0) \right\|_{X_{b,d,r}^{s-1,\varepsilon}} \\
 & \leq C_3 \left\| (\eta_0, u_0) \right\|_{E_{b,d,r}^{s,\varepsilon}},
 \end{aligned}$$

where we used (3.6), (3.7) and (3.10).

Proving the uniqueness of the solution is done in the following lines. Let us suppose that  $(\bar{\eta}^1, \bar{u}^1), (\bar{\eta}^2, \bar{u}^2) \in \mathcal{C} \left( [0, T], E_{b,d,r}^s \right)$  are two solutions of (3.1). Then, we observe that:

$$\begin{cases} (I - \varepsilon b \partial_{xx}^2) \partial_t \delta \bar{\eta} + \partial_x \delta \bar{u} + \varepsilon \partial_x (\bar{u}^1 \delta \bar{\eta} + \bar{\eta}^2 \delta \bar{u}) = 0, \\ (I - \varepsilon d \partial_{xx}^2) \partial_t \delta \bar{u} + \partial_x \delta \bar{\eta} + \varepsilon \bar{u}^1 \partial_x \delta \bar{u} + \varepsilon \delta \bar{u} \partial_x \bar{u}^2 = 0, \\ \delta \bar{\eta}|_{t=0} = 0, \quad \delta \bar{u}|_{t=0} = 0. \end{cases} \tag{3.11}$$

Writing the first equation of (3.11) in integral form we get that

$$\delta \bar{\eta}(t) = \int_0^t (I - \varepsilon b \partial_{xx}^2)^{-1} (\partial_x \delta \bar{u} + \delta \bar{\eta} \partial_x \bar{u}^1 + \bar{u}^1 \partial_x \delta \bar{\eta} + \bar{\eta}^2 \partial_x \delta \bar{u} + \delta \bar{u} \partial_x \bar{\eta}^2) d\tau$$

and because

$$\partial_x \delta \bar{u} + \delta \bar{\eta} \partial_x \bar{u}^1 + \bar{u}^1 \partial_x \delta \bar{\eta} + \bar{\eta}^2 \partial_x \delta \bar{u} + \delta \bar{u} \partial_x \bar{\eta}^2 \in L^2(\mathbb{R})$$

we get that

$$\delta \bar{\eta} \in H^{2 \operatorname{sgn}(b)}(\mathbb{R}).$$

Similarly

$$\delta \bar{u} \in H^{2 \operatorname{sgn}(d)}(\mathbb{R}).$$

Thus, multiplying the first equation of (3.11) with  $\delta \bar{\eta}$  and the second one with  $\delta \bar{u}$  and using Gronwall's lemma will lead us to the conclusion  $(\delta \bar{\eta}, \delta \bar{u}) = (0, 0)$  on  $[0, T]$ . This ends the proof of Theorem 1.1.

### 3.2 The 2d case: proof of Theorem 1.2

For the reader's convenience, let us rewrite below the problem that we wish to solve:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \bar{\eta} + \operatorname{div} \bar{V} + \varepsilon \operatorname{div} (\bar{\eta} \bar{V}) = 0, \\ (I - \varepsilon d \Delta) \partial_t \bar{V} + \nabla \bar{\eta} + \varepsilon \bar{V} \cdot \nabla \bar{V} = 0, \\ \bar{\eta}|_{t=0}(x, y) = \eta_0(x) + \phi(x, y), \\ \bar{V}|_{t=0}(x, y) = (u_0(x), 0) + \psi(x, y). \end{cases} \quad (3.12)$$

According to Theorem 1.1, there exist  $\varepsilon_0, C^{1D} > 0$ , depending on  $\sigma, b, d$  and the norm of the initial data  $(\eta_0, u_0)$  such that for any  $\varepsilon \leq \varepsilon_0$  we may consider  $(\eta^{\varepsilon, 1D}, u^{\varepsilon, 1D}) \in \mathcal{C}\left([0, \frac{C^{1D}}{\varepsilon}], E_{b,d,r}^\sigma(\mathbb{R})\right)$  the unique solution of the 1-dimensional system (3.1) with initial data  $(\eta_0, u_0)$  and

$$\sup_{t \in [0, \frac{C^{1D}}{\varepsilon}]} \|(\eta^{\varepsilon, 1D}(t), u^{\varepsilon, 1D}(t))\|_{E_{b,d,r}^{\sigma, \varepsilon}} + \sup_{t \in [0, \frac{C^{1D}}{\varepsilon}]} \|\partial_t \eta^{\varepsilon, 1D}(t)\|_{L^\infty} \leq \tilde{C}_1 \|(\eta_0, u_0)\|_{E_{b,d,r}^{\sigma, \varepsilon}}, \quad (3.13)$$

with  $\tilde{C}_1$  some numerical constant. Let us observe that if we consider

$$\begin{cases} \eta_1^\varepsilon(t, x, y) = \eta^{1D}(t, x), \\ V_1^\varepsilon(t, x, y) = (u^{1D}(t, x), 0), \end{cases}$$

for all  $(t, x, y) \in [0, \frac{C^{1D}}{\varepsilon}] \times \mathbb{R}^2$ , then:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta_1^\varepsilon + \operatorname{div} V_1^\varepsilon + \varepsilon \operatorname{div} (\eta_1^\varepsilon V_1^\varepsilon) = 0, \\ (I - \varepsilon d \Delta) \partial_t V_1^\varepsilon + \nabla \eta_1^\varepsilon + \varepsilon V_1^\varepsilon \cdot \nabla V_1^\varepsilon = 0, \\ \eta_1^\varepsilon|_{t=0}(x, y) = \eta_0(x), \quad V_1^\varepsilon|_{t=0} = (u_0(x), 0). \end{cases}$$

Observe that we have  $\partial_x \eta^{\varepsilon, 1D}(t) \in B_{2,r}^{\sigma-1}(\mathbb{R}) \hookrightarrow B_{\infty, \infty}^{\sigma-\frac{3}{2}}(\mathbb{R})$ . Let us observe that for all  $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$  such that

$$|x_1 - x_2|^2 + |y_1 - y_2|^2 \leq 1$$

the following holds true

$$\begin{aligned} |\nabla \eta_1^\varepsilon(t, x_1, y_1) - \nabla \eta_1^\varepsilon(t, x_2, y_2)| &= |\partial_x \eta^{\varepsilon, 1D}(t, x_1) - \partial_x \eta^{\varepsilon, 1D}(t, x_2)| \\ &\leq C \|\partial_x \eta^{\varepsilon, 1D}(t)\|_{B_{\infty, \infty}^{\sigma-\frac{3}{2}}(\mathbb{R})} |x_1 - x_2|^{(\sigma-\frac{3}{2}) - [\sigma-\frac{3}{2}]}. \end{aligned}$$

Using the fact that  $B_{\infty, \infty}^{\sigma-\frac{3}{2}}(\mathbb{R}^2) = \mathcal{C}^{\sigma-\frac{3}{2}, \sigma-\frac{3}{2}-[\sigma-\frac{3}{2}]}(\mathbb{R}^2)$ , see Remark 1.2 and relation (1.13), we get that  $\nabla \eta_1^\varepsilon(t) = (\partial_x \eta^{\varepsilon, 1D}(t), 0) \in B_{\infty, r}^s(\mathbb{R}^2)$  with

$$\|\nabla \eta_1^\varepsilon(t)\|_{B_{\infty, r}^s(\mathbb{R}^2)} \leq C \|\partial_x \eta^{\varepsilon, 1D}(t)\|_{B_{2,r}^{\sigma-1}(\mathbb{R})}. \quad (3.14)$$

Similarly, we get:

$$\|\nabla V_1^\varepsilon\|_{B_{\infty, r}^s(\mathbb{R}^2)} \leq C \|\partial_x u^{\varepsilon, 1D}\|_{B_{2,r}^{\sigma-1}(\mathbb{R})}. \quad (3.15)$$

It is also clear that:

$$\|(\eta_1^\varepsilon, V_1^\varepsilon)\|_{L^\infty(\mathbb{R}^2)} = \|(\eta^{\varepsilon, 1D}, u^{\varepsilon, 1D})\|_{L^\infty(\mathbb{R})} \quad \text{and} \quad \|\partial_t \eta_1^\varepsilon\|_{L^\infty(\mathbb{R}^2)} = \|\partial_t \eta^{\varepsilon, 1D}\|_{L^\infty(\mathbb{R})}, \quad (3.16)$$

and thus, using (3.14), (3.15), (3.16) respectively the uniform bounds of (3.13), we get that:

$$\begin{aligned} \sup_{\varepsilon \in [0, \varepsilon_0]} \sup_{t \in [0, \frac{C^{1D}}{\varepsilon}]} \left( \|(\eta_1^\varepsilon(t), V_1^\varepsilon(t))\|_{L^\infty(\mathbb{R}^2)} + \|(\nabla \eta_1^\varepsilon(t), \nabla V_1^\varepsilon(t))\|_{B_{\infty, r}^s(\mathbb{R}^2)} \right) + \sup_{t \in [0, \frac{C^{1D}}{\varepsilon}]} \|\partial_t \eta_1^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2)} \\ \leq \tilde{C}_2 \|(\eta_0, u_0)\|_{E_{b,d,r}^{s,1}}. \end{aligned} \quad (3.17)$$

Let us consider the system:

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \eta + \operatorname{div} V + \varepsilon \operatorname{div} (\eta V_1^\varepsilon + \eta_1^\varepsilon V + \eta V) = 0, \\ (I - \varepsilon d \Delta) \partial_t V + \nabla \eta + \varepsilon (V_1^\varepsilon + V) \cdot \nabla V + \varepsilon V \cdot \nabla V_1^\varepsilon = 0, \\ \eta|_{t=0} = \phi, \quad V|_{t=0} = \psi. \end{cases}$$

Using the uniform bounds of  $(\eta_1^\varepsilon, V_1^\varepsilon)_{\varepsilon \leq \varepsilon_0}$  and owing to Theorem 2.2, there exist two real numbers  $C^{2D} \leq C^{1D}$ ,  $\varepsilon_1 \leq \varepsilon_0$  depending on  $s, \sigma, b, d$  and on  $\|(\eta_0, u_0)\|_{E_{b,d,r}^{s,1}} + \|(\phi, \psi)\|_{X_{b,d,r}^{s,1}}$  such that for any  $\varepsilon \leq \varepsilon_1$  we can uniquely construct  $(\eta^{\varepsilon, 2D}, V^{\varepsilon, 2D}) \in \mathcal{C}\left([0, \frac{C^{2D}}{\varepsilon}], X_{b,d,r}^s\right)$  and, moreover,

$$\sup_{t \in [0, \frac{C^{2D}}{\varepsilon}]} \|(\eta^{\varepsilon, 2D}(t), V^{\varepsilon, 2D}(t))\|_{X_{b,d,r}^{s,\varepsilon}} \leq \tilde{C}_3 \left( \|(\eta_0, u_0)\|_{E_{b,d,r}^{s,\varepsilon}} + \|(\phi, \psi)\|_{X_{b,d,r}^{s,\varepsilon}} \right) \quad (3.18)$$

where  $\tilde{C}_3$  is numerical constant. Then, we see that defining

$$\begin{cases} \bar{\eta}^\varepsilon(t, x, y) = \eta^{\varepsilon, 1D}(t, x) + \eta^{\varepsilon, 2D}(t, x, y), \\ \bar{V}^\varepsilon(t, x, y) = (u^{\varepsilon, 1D}(t, x), 0) + V^{\varepsilon, 2D}(t, x, y) \end{cases}$$

for all  $(t, x, y) \in [0, \frac{C^{2D}}{\varepsilon}] \times \mathbb{R}^2$  is a solution of (1.7) which, by construction, belongs to  $M_{b,d,r}^{\sigma,s}$ . Moreover, owing to (3.13) and (3.18) there exists a constant  $\tilde{C}_4$  such that:

$$\|(\bar{\eta}^\varepsilon, \bar{V}^\varepsilon)\|_{M_{b,d,r}^{\sigma,s,\varepsilon}} \leq \tilde{C}_4 \left( \|(\eta_0, u_0)\|_{E_{b,d,r}^{s,\varepsilon}} + \|(\phi, \psi)\|_{X_{b,d,r}^{s,\varepsilon}} \right).$$

We proceed by proving the uniqueness property. Consider two solutions  $(\eta^i + \phi^i, (u^i + \psi_1^i, \psi_2^i)) \in \mathcal{C}([0, T]; M_{b,d,r}^{\sigma,s})$  with the same initial data and we introduce the following notations:

$$\begin{cases} \delta \eta = \eta^1 - \eta^2, \delta u = u^1 - u^2, \\ \delta \phi = \phi^1 - \phi^2, \delta \psi = \psi^1 - \psi^2. \end{cases}$$

For any pair  $\rho \in \mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}^2))$  we get that:

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^2} (\delta \eta + \delta \phi) (I - \varepsilon b \Delta) \partial_t \rho + \int_0^t \int_{\mathbb{R}^2} (\varepsilon (\delta \eta + \delta \phi) (u^1 + \psi_1^1) + (1 + \varepsilon (\eta^2 + \phi^2)) (\delta u + \delta \psi_1)) \partial_x \rho \\ & + \int_0^t \int_{\mathbb{R}^2} (\varepsilon (\delta \eta + \delta \phi) \psi_2^1 + (1 + \varepsilon (\eta^2 + \phi^2)) \delta \psi_2) \partial_y \rho - \int_{\mathbb{R}^2} (\delta \eta(t) + \delta \phi(t)) (I - \varepsilon b \Delta) \rho(t) = 0. \end{aligned} \quad (3.19)$$

Taking for any  $m \in \mathbb{N}^*$

$$\rho^m(t, x, y) = \frac{1}{m} \tilde{\rho}(t, x) \chi\left(\frac{y}{m}\right)$$

with  $\tilde{\rho} \in \mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}))$  and  $\chi \in \mathcal{S}(\mathbb{R})$  we find that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \delta \eta (I - \varepsilon b \partial_{xx}^2) \partial_t \tilde{\rho} + \int_0^t \int_{\mathbb{R}} (\varepsilon \delta \eta u^1 + (1 + \varepsilon \eta^2) \delta u) \partial_x \tilde{\rho} \\ & - \int_{\mathbb{R}} \delta \eta(t) (I - \varepsilon b \partial_{xx}^2) \tilde{\rho}(t) = o(m) \quad \text{when } m \rightarrow \infty. \end{aligned}$$

Proceeding similarly we get that for any  $\mu \in \mathcal{C}([0, T]; \mathcal{S}(\mathbb{R}))$

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} \delta u (I - \varepsilon d \partial_{xx}^2) \partial_t \mu + \int_0^t \int_{\mathbb{R}} (\delta \eta + \varepsilon u^1 \delta u) \partial_x \mu + \int_0^t \int_{\mathbb{R}} \varepsilon \delta u (\partial_x u^1 - \partial_x u^2) \mu \\ & - \int_{\mathbb{R}} \delta u(t) (I - \varepsilon d \partial_{xx}^2) \mu(t) = o(m) \quad \text{when } m \rightarrow \infty. \end{aligned}$$

We thus recover that the pair  $(\delta\eta, \delta u) \in \mathcal{C}\left([0, T]; E_{b,d,r}^\sigma(\mathbb{R})\right)$  is a solution of

$$\begin{cases} (I - \varepsilon b \partial_{xx}) \partial_t \delta\eta + \partial_x \delta u + \varepsilon \partial_x (\delta\eta u^1 + \eta^2 \delta u) = 0, \\ (I - \varepsilon d \partial_{xx}) \partial_t \delta u + \partial_x \eta + \varepsilon u^1 \partial_x \delta u + \varepsilon \delta u \partial_x u^2 = 0, \\ \delta\eta|_{t=0} = 0, \quad \delta u|_{t=0} = 0. \end{cases} \quad (3.20)$$

According to Theorem 1.1 this implies that  $(\delta\eta, \delta u) = (0, 0)$ . It follows that

$$\begin{cases} (I - \varepsilon b \Delta) \partial_t \delta\phi + \operatorname{div} \delta\psi + \varepsilon \operatorname{div} (\delta\phi \psi^1 + \phi^2 \delta\psi) = 0, \\ (I - \varepsilon d \Delta) \partial_t \delta\psi + \nabla \delta\phi + \varepsilon \psi^1 \cdot \nabla \delta\psi + \varepsilon \delta\psi \cdot \nabla \psi^2 = 0, \\ \delta\phi|_{t=0} = 0, \quad \delta\psi|_{t=0} = 0, \end{cases} \quad (3.21)$$

which by Theorem 2.1 implies that  $(\delta\phi, \delta\psi) = (0, 0)$ . Thus we obtain that the two solutions coincide.

**Remark 3.1.** In the case when  $b > 0$  and  $d > 0$  a stronger result holds in the sense that one may prove stability estimates in  $L^\infty(\mathbb{R}^2) \times (L^\infty(\mathbb{R}^2))^2$ . This is done by observing that for  $l = 1, 2$  we have:

$$\mathcal{F}^{-1} \left( \frac{\xi_l}{1 + |\xi|^2} \right) \in L^1(\mathbb{R}^2)$$

see for instance [6], page 606.

## 4 Appendix: Littlewood-Paley theory

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [3]. In the following if  $\Omega \subset \mathbb{R}^n$  is a domain then  $\mathcal{D}(\Omega)$  will denote the set of smooth functions on  $\Omega$  with compact support and  $\mathcal{S}$  will denote the Schwartz class of functions defined on  $\mathbb{R}^n$ . Also, we consider  $\mathcal{S}'$  the set of tempered distributions on  $\mathbb{R}^n$ .

### 4.1 The dyadic partition of unity

Let us begin by recalling the so called Bernstein lemma:

**Lemma 4.1.** Let  $\mathcal{C}$  be a given annulus and  $B$  a ball of  $\mathbb{R}^n$ . Let us also consider any nonnegative integer  $k$ , a couple  $p, q \in [1, \infty]^2$  with  $p \leq q$  and any functions  $u, v \in L^p$  such that  $\operatorname{Supp}(\hat{u}) \subset \lambda B$  and  $\operatorname{Supp}(\hat{v}) \subset \lambda \mathcal{C}$ . Then, there exists a constant  $C$  such that the following inequalities hold true:

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+n(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}, \quad (4.1)$$

respectively

$$C^{-k-1} \lambda^k \|v\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha v\|_{L^p} \leq C^{k+1} \lambda^k \|v\|_{L^p}. \quad (4.2)$$

Next, let us introduce the dyadic partition of the space:

**Proposition 4.1.** Let  $\mathcal{C}$  be the annulus  $\{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$ . There exist two radial functions  $\chi \in \mathcal{D}(B(0, 4/3))$  and  $\varphi \in \mathcal{D}(\mathcal{C})$  valued in the interval  $[0, 1]$  and such that:

$$\forall \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad (4.3)$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad (4.4)$$

$$2 \leq |j - j'| \Rightarrow \operatorname{Supp}(\varphi(2^{-j} \cdot)) \cap \operatorname{Supp}(\varphi(2^{-j'} \cdot)) = \emptyset \quad (4.5)$$

$$j \geq 1 \Rightarrow \operatorname{Supp}(\chi) \cap \operatorname{Supp}(\varphi(2^{-j} \cdot)) = \emptyset \quad (4.6)$$

the set  $\tilde{\mathcal{C}} = \mathcal{B}(0, 2/3) + \mathcal{C}$  is an annulus and we have

$$|j - j'| \geq 5 \Rightarrow 2^j \mathcal{C} \cap 2^{j'} \tilde{\mathcal{C}} = \emptyset. \quad (4.7)$$

Also the following inequalities hold true:

$$\forall \xi \in \mathbb{R}^n, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1, \quad (4.8)$$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1. \quad (4.9)$$

From now on we fix two functions  $\chi$  and  $\varphi$  satisfying the assertions of Proposition 4.1 and let us denote by  $h$  respectively  $\tilde{h}$  their Fourier inverses. The following lemma shows how to reconstruct a tempered distribution once we know its nonhomogeneous dyadic blocks (see (1.8)).

**Lemma 4.2.** For any  $u \in \mathcal{S}'$  we have:

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

## 4.2 Properties of Besov spaces

The following propositions list important basic properties of Besov spaces that are used through the paper.

**Proposition 4.2.** A tempered distribution  $u \in \mathcal{S}'$  belongs to  $B_{p,r}^s$  if and only if there exists a sequence  $(c_j)_j$  such that  $(2^{js} c_j)_j \in \ell^r(\mathbb{Z})$  with norm 1 and a constant  $C(u) > 0$  such that for any  $j \in \mathbb{Z}$  we have

$$\|\Delta_j u\|_{L^p} \leq C(u) c_j.$$

**Proposition 4.3.** Let  $s, \tilde{s} \in \mathbb{R}$  and  $r, \tilde{r} \in [1, \infty]$ .

- $B_{p,r}^s$  is a Banach space which is continuously embedded in  $\mathcal{S}'$ .
- The inclusion  $B_{p,r}^s \subset B_{p,\tilde{r}}^{\tilde{s}}$  is continuous whenever  $\tilde{s} < s$  or  $s = \tilde{s}$  and  $\tilde{r} > r$ .
- We have the following continuous inclusion  $B_{p,1}^{\frac{n}{p}} \subset \mathcal{C}_0^5(\subset L^\infty)$ .
- (Fatou property) If  $(u_n)_{n \in \mathbb{N}}$  is a bounded sequence of  $B_{p,r}^s$  which tends to  $u$  in  $\mathcal{S}'$  then  $u \in B_{p,r}^s$  and

$$\|u\|_{B_{p,r}^s} \leq \liminf_n \|u_n\|_{B_{p,r}^s}.$$

- If  $r < \infty$  then

$$\lim_{j \rightarrow \infty} \|u - S_j u\|_{B_{p,r}^s} = 0.$$

**Proposition 4.4.** Let us consider  $m \in \mathbb{R}$  and a smooth function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that for all multi-index  $\alpha$ , there exists a constant  $C_\alpha$  such that:

$$\forall \xi \in \mathbb{R}^n \quad |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m - |\alpha|}.$$

Then the operator  $f(D)$  is continuous from  $B_{p,r}^s$  to  $B_{p,r}^{s-m}$ .

**Proposition 4.5.** Let  $1 \leq r \leq \infty$ ,  $s \in \mathbb{R}$  and  $\varepsilon > 0$ . For all  $\phi \in \mathcal{S}$ , the map  $u \rightarrow \phi u$  is compact from  $B_{2,r}^s$  to  $B_{2,r}^{s-\varepsilon}$ .

The next result deals with product estimates in nonhomogeneous Besov spaces.

**Theorem 4.1.** A constant  $C$  exists such that the following holds true. Consider a real number  $s > 0$  and any  $(p, p_1, p_2, p_3, p_4, r) \in [1, \infty]^6$  such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

For any  $(u, v) \in L^{p_2} \times (L^{p_4} \cap B_{p_1,r}^s)$  such that  $\nabla u \in B_{p_3,r}^{s-1}$  we have:

$$\|uv\|_{B_{p,r}^s} \leq \|u\|_{L^{p_2}} \|v\|_{B_{p_1,r}^s} + \|\nabla u\|_{B_{p_3,r}^{s-1}} \|v\|_{L^{p_4}}.$$

All the above statements are given or immediate consequences of the results presented in [3], pages 107 – 108.

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<sup>5</sup> $\mathcal{C}_0$  is the space of continuous bounded functions which decay at infinity.

### 4.3 Commutator estimates

This section is devoted to establish commutator-type estimates both in homogeneous and in nonhomogeneous Besov spaces. We begin by stating the following basic, yet very useful lemma:

**Lemma 4.3.** Let us consider  $\theta$  a  $\mathcal{C}^1$  function on  $\mathbb{R}^n$  such that  $(1 + |\cdot|)\hat{\theta} \in L^1$ . Let us also consider  $p, q \in [1, \infty]$  such that:

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \leq 1.$$

Then, there exists a constant  $C$  such that for any Lipschitz function  $a$  with gradient in  $L^p$ , any function  $b \in L^q$  and any positive  $\lambda$ :

$$\|[\theta(\lambda^{-1}D), a]b\|_{L^r} \leq C\lambda^{-1} \|\nabla a\|_{L^p} \|b\|_{L^q}.$$

In particular, when  $\theta = \varphi$  and  $\lambda = 2^j$  we get that:

$$\|[\Delta_j, a]b\|_{L^r} \leq C2^{-j} \|\nabla a\|_{L^p} \|b\|_{L^q}.$$

**Proposition 4.6.** Let us consider  $s > 0$  and  $(p, p_1, r) \in [1, \infty]^3$  such that  $p \leq p_1$ . Also, let us consider two tempered distributions  $(u, v)$  such that  $u \in B_{\infty, \infty}^M(\mathbb{R}^n)$ ,  $\nabla u \in L^\infty(\mathbb{R}^n) \cap B_{p_1, r}^{s-1}(\mathbb{R}^n)$  and  $\nabla v \in L^{p_2}(\mathbb{R}^n) \cap B_{p_1, r}^s(\mathbb{R}^n)$  where  $M$  is some strictly negative real number and  $\frac{1}{p_2} = \frac{1}{p} - \frac{1}{p_1}$ . We denote by

$$R_j = [\Delta_j, u]\partial^\alpha v = \Delta_j(u\partial^\alpha v) - u\Delta_j\partial^\alpha v,$$

where  $\alpha \in \overline{1, n}$ . Then, the following estimate holds true:

$$\|(2^{js} \|R_j\|_{L^p})\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla u\|_{L^\infty} \|\nabla v\|_{B_{p, r}^{s-1}} + \|\nabla v\|_{L^{p_2}} \|\nabla u\|_{B_{p_1, r}^{s-1}}.$$

*Proof.* The main ingredients of the proof are Lemma 4.3 and the Bony decomposition of the product into the paraproduct and remainder. For the definitions and some properties we refer to [3] pages 106-108. We begin by considering

$$\tilde{u} = u - \dot{S}_{-1}u$$

and we write that

$$R_j = \sum_{i=1,6} R_j^i$$

where

$$\begin{cases} R_j^1 = \Delta_j(T_{\tilde{u}}\partial^\alpha v) - T_{\tilde{u}}\Delta_j\partial^\alpha v, & R_j^2 = \Delta_j(T_{\partial^\alpha v}\tilde{u}), \\ R_j^3 = T_{\Delta_j\partial^\alpha v}\tilde{u}, & R_j^4 = \Delta_j R(\tilde{u}, \partial^\alpha v), \\ R_j^5 = R(\tilde{u}, \Delta_j\partial^\alpha v). & R_j^6 = [\Delta_j, S_{-1}u]\partial^\alpha v. \end{cases}$$

We begin with  $R_j^1$ . We notice that  $(2^{lM} \|S_{l-1}\tilde{u}\Delta_l\partial^\alpha v\|_{L^\infty})_{l \geq -1} \in \ell^\infty$  and

$$\|(2^{lM} \|S_{l-1}\tilde{u}\Delta_l\partial^\alpha v\|_{L^\infty})_{l \geq -1}\|_{\ell^\infty} \leq C \|u\|_{B_{\infty, \infty}^M} \|\nabla v\|_{L^\infty}$$

such that the series  $\sum_{l \geq -1} S_{l-1}u\Delta_l\partial^\alpha v$  is convergent. Thus, owing to Lemma 4.3 we may write that

$$2^{js} \|R_j^1\|_{L^p} \leq \sum_{|j-l| \leq 4} 2^{(j-l)s} 2^{ls} \|[\Delta_j, S_{l-1}\tilde{u}]\Delta_l\partial^\alpha v\|_{L^p} \leq C \|\nabla u\|_{L^\infty} \sum_{|j-l| \leq 4} 2^{(j-l)(s-1)} 2^{l(s-1)} \|\Delta_l \nabla v\|_{L^p}.$$

From Young's inequality we conclude that

$$\|(2^{js} \|R_j^1\|_{L^p})\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla u\|_{L^\infty} \|\nabla v\|_{B_{p, r}^{s-1}}.$$

Let us pass to  $R_j^2$ . We have that:

$$2^{js} \|R_j^2\|_{L^p} \leq \sum_{|j-l| \leq 4} 2^{(j-l)s} 2^{ls} \|S_{l-1}\partial^\alpha v\Delta_l\tilde{u}\|_{L^p} \leq C \|\nabla v\|_{L^{p_2}} \sum_{|j-l| \leq 4} 2^{(j-l)s} 2^{ls} \|\Delta_l\tilde{u}\|_{L^{p_1}}$$

$$\leq C \|\nabla v\|_{L^{p_2}} \sum_{|j-l| \leq 4} 2^{(j-l)s} 2^{l(s-1)} \|\Delta_l \nabla \tilde{u}\|_{L^{p_1}}.$$

From Young's inequality we conclude that

$$\left\| \left( 2^{js} \|R_j^2\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla v\|_{L^{p_2}} \|\nabla u\|_{B_{p_1,r}^{s-1}}.$$

The third term is treated similar:

$$2^{js} \|R_j^3\|_{L^p} \leq \sum_{j-l \leq 5} 2^{(j-l)s} 2^{ls} \|S_{l-1} \partial^\alpha v \Delta_l \tilde{u}\|_{L^p} \leq C \|\nabla v\|_{L^{p_2}} \sum_{j-l \leq 5} 2^{(j-l)s} 2^{l(s-1)} \|\Delta_l \nabla \tilde{u}\|_{L^{p_1}}.$$

Because  $s > 0$  we can apply Young's inequality and obtain that

$$\left\| \left( 2^{js} \|R_j^3\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla v\|_{L^{p_2}} \|\nabla u\|_{B_{p_1,r}^{s-1}}.$$

As for the fourth term

$$2^{js} \|R_j^4\|_{L^p} \leq \sum_{j-l \leq 5} 2^{(j-l)s} 2^{ls} \|\Delta_l \tilde{u}\|_{L^\infty} \left\| \tilde{\Delta}_l \partial^\alpha v \right\|_{L^p} \leq C \sum_{j-l \leq 5} 2^{(j-l)s} 2^{l(s-1)} \|\Delta_l \nabla \tilde{u}\|_{L^\infty} \left\| \tilde{\Delta}_l \partial^\alpha v \right\|_{L^p}.$$

Because  $s > 0$  we can apply Young's inequality and obtain that:

$$\left\| \left( 2^{js} \|R_j^4\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla u\|_{L^\infty} \|\nabla v\|_{B_{p,r}^{s-1}}.$$

In a similar manner we get that:

$$\left\| \left( 2^{js} \|R_j^5\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla u\|_{L^\infty} \|\nabla v\|_{B_{p,r}^{s-1}}.$$

Finally, the sixth term is treated as it follows. First we write that there exists a integer  $N_0$  such that:

$$\left[ \Delta_j, \dot{S}_{-1} u \right] \partial^\alpha v = \sum_{l \geq -1} \left[ \Delta_j, \dot{S}_{-1} u \right] \Delta_l \partial^\alpha v = \sum_{|l-j| \leq N_0} \left[ \Delta_j, \dot{S}_{-1} u \right] \Delta_l \partial^\alpha v.$$

This comes from:

$$\text{Supp} \left( \dot{S}_{-1} u \Delta_l \partial^\alpha v \right) \subset 2^j \mathcal{C} \cap \left( B \left( 0, \frac{2}{3} \right) + 2^l \mathcal{C} \right) \subset 2^j \mathcal{C} \cap 2^l \left( B \left( 0, \frac{2}{3} \right) + \mathcal{C} \right)$$

and from the fact that  $B \left( 0, \frac{2}{3} \right) + \mathcal{C}$  is an annulus. Thus from Lemma 4.3 we get that :

$$2^{js} \|R_j^6\|_{L^p} \leq \sum 2^{(j-l)(s-1)} 2^{l(s-1)} \left\| \nabla \dot{S}_{-1} u \right\|_{L^\infty} \|\Delta_l \partial^\alpha v\|_{L^p}$$

and consequently

$$\left\| \left( 2^{js} \|R_j^6\|_{L^p} \right) \right\|_{\ell^r(\mathbb{Z})} \lesssim_s \|\nabla u\|_{L^\infty} \|\nabla v\|_{B_{p,r}^{s-1}}.$$

By putting together the estimates for  $R_j^1, R_j^2, R_j^3, R_j^4, R_j^5, R_j^6$ , we end the proof of Proposition 4.6.  $\square$

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